Financing the Scale-Up: Equity, Innovation and Economic Growth*

Andrea Guccione† Bank of Italy

Juan Segura‡ CEMFI

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Abstract

Innovative firms rely on external finance to fund their operations, yet debt and equity may play distinct roles in shaping this process. Using balance-sheet and patent-level data on European patenting firms, we document two facts. First, external equity flows to small, young, collateral-poor, faster-growing firms, while debt reaches larger, more collateral-rich firms. Second, real outcomes, including innovation, respond positively to capital inflows, and these responses are substantially larger for external equity. We interpret these facts through a Schumpeterian model of innovation-led growth with financial frictions in which heterogeneous firms finance their operations with retained earnings, debt, and external equity. In the model, financially constrained firms invest less in R&D, because successful innovations are worth less when capital cannot be scaled to exploit them. This gives rise to selection: equity reaches the firms where new capital has the highest return and its effect on innovation is largest. Financial frictions also depress firm entry, further reducing growth. The calibrated model matches the dynamic correlation of patenting to capital injections and its heterogeneity across financing sources. We use the model to quantify the aggregate effects of financial frictions and evaluate growth policies. In the perfect credit benchmark, growth is 1.25 percentage points higher than in the baseline; frictionless equity markets recover 13% of this gap. Government subsidies to external equity deliver the largest welfare gains at low levels of government spending, while R&D subsidies become more effective as spending rises.

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†Via Nazionale 91, 00184, Rome, Italy. Email: andrea.guccione@bancaditalia.it.

‡CEMFI address: Casado del Alisal 5, 28014 Madrid, Spain. Email: juan.segura@cemfi.edu.es.

1 Introduction

Over the past two decades, Europe has experienced sluggish productivity growth alongside a widening gap relative to the United States. Despite a strong scientific base and an active startup ecosystem, few European firms transition into large-scale global innovators. In fact, over the past fifty years, Europe has produced only a handful of firms with market capitalizations above €10 billion, and no EU-based company founded during this period has reached a valuation above €100 billion (Draghi, 2024). A central concern is therefore not the creation of new firms, but their ability to scale-up.

A prominent explanation for this scale-up gap emphasizes access to external finance. The expansion of innovative firms is capital intensive and characterized by sustained small or negative cash flows, requiring substantial external financing. Yet not all external finance is alike: debt and equity may reach very different firms. Whether this distinction matters for aggregate growth, and how effective finance-side policies are at stimulating it, remains open.

We begin by providing reduced-form evidence consistent with external finance shaping firm-level innovation and growth dynamics. Using firm-level data for the four largest continental European economies from ORBIS, we first show that external equity and debt fund different firms: external-equity inflows are systematically directed toward firms that are smaller, younger, hold less net worth and collateral, and grow faster. We then document that increases in financial capital are followed by persistent increases in innovative output over subsequent years. This relationship is heterogeneous across funding sources: a 10% increase in financial capital from external-equity injections is associated with 0.26 additional patent applications over the next four years, whereas comparable increases from debt and internal funds have significantly smaller effects. Capital inflows are also accompanied by immediate and sustained increases in tangible and intangible investment and sales, particularly when driven by external equity. Moreover, these patterns are stronger for small and young firms.

Motivated by these findings, we develop a Schumpeterian endogenous growth model with frictional debt and equity markets. The model economy consists of many sectors. In each sector, firms produce perfectly substitutable varieties of the same good using capital and labor, and compete in prices à la Bertrand for market leadership. Firms are multi-product and grow in the product space by investing in R&D. Successful innovations allow firms to capture new product lines through creative destruction. To finance their operations, firms rely on internal funds or costly external financing from two sources: (i) borrowing, which is limited by a collateral constraint; and (ii) equity injections, that require infrequent

meetings with outside investors, governed by an exogenous matching rate. In case of a meeting, if the surplus from the transaction is positive, outside investors make an equity injection in exchange for acquiring an ownership share in the firm.

In our model, financial constraints shape firms' innovation choices. Firms with little net worth cannot produce at their optimal scale, which reduces their current profits. This generates an incentive to accumulate net worth to raise production capacity, increasing constrained firms' opportunity cost of R&D investments. At the same time, the benefits from innovating are lower for these firms, because acquiring a new product is less valuable when capital cannot be scaled up to fully exploit it. Together, these forces imply that financial constraints reduce the propensity to innovate for firms with low net worth per product line. Because the surplus from an equity injection is larger for firms with a higher return to net worth, equity flows disproportionately to these constrained, high-return firms. Firms are otherwise indifferent to the source of their net worth, so equity's distinctive role arises entirely from this selection. Moreover, financing constraints also depress the value of firm creation, further reducing the aggregate rate of creative destruction.

We discipline the model using aggregate and micro-level moments on firm dynamics and finance for the French economy. The resulting economy reproduces how leveraged firms are across the size distribution, as well as the frequency and size of equity injections observed in the data. In addition, the calibrated model matches the dynamic response of patenting to capital injections, as well as the heterogeneity across financing sources.

We then use our model to quantify how financial constraints affect firms' innovation and scaling decisions, and how these distortions translate into dynamic output losses at the aggregate level. Easing financial constraints boosts aggregate productivity growth by stimulating entry and raising the R&D investment of financially constrained incumbents. In the perfect credit benchmark, the growth rate of the economy is 1.25 percentage points larger than in the baseline, primarily driven by higher incumbent innovation. Frictionless equity markets would close roughly 13% of the gap, also by stimulating higher R&D investments by incumbents. Conversely, shutting equity markets down lowers growth by 25 basis points with respect to the baseline. Search frictions or lack of investors, rather than issuance costs, emerge as the main market imperfection limiting the role of equity finance.

Finally, we use the model to evaluate policies aimed at stimulating innovation and growth, comparing subsidies to debt, external equity, and R&D. All three policy instruments increase growth and welfare, but they operate through different margins. At low levels of government spending, the equity subsidy generates the largest welfare gains, because it directs resources toward firms with a high shadow value of funds. As the subsidy expands and equity is increasingly allocated to firms with lower returns to additional

funding, the R&D subsidy becomes more effective, while the debt subsidy remains the least effective instrument throughout. When equity and R&D subsidies are combined, the welfare-maximizing policy allocates spending exclusively to external equity at low government spending levels, then shifts progressively toward R&D as total spending increases. At government spending equal to 1% of GDP, the optimal mix subsidizes R&D at 26.3% and equity at 8.8%, and leads to an 11.3% gain in consumption-equivalent welfare.

Related literature In this paper, we bridge two strands of the literature. The first is the literature on endogenous growth, firm and industry dynamics (Klette and Kortum, 2004; Lentz and Mortensen, 2008; Akcigit and Kerr, 2018; Acemoglu, Akcigit, Alp, Bloom and Kerr, 2018; Peters, 2020). This literature studies how firms' endogenous innovation decisions translate into aggregate productivity growth but typically abstracts from firms' financing choices: the financing structure of the firm is left unspecified, and innovation is implicitly financed out of an unconstrained pool of internal funds. Our contribution is to explicitly model firms' capital structure decisions, to study how the financing structure of a firm shapes its productivity-enhancing investments, and how limits in access to external finance affect firms' innovation decisions. At the macro level, we study how easing firms' external financing constraints impacts economic growth.

The second literature we contribute to is the extensive body of research studying the effects of financial frictions on firm dynamics (Kiyotaki and Moore, 1997; Jeong and Townsend, 2007; Buera, 2009; Buera, Kaboski and Shin, 2011; Midrigan and Xu, 2014; Moll, 2014; Peter, 2021). This literature has emphasized the role of financial frictions in distorting the allocation of inputs across firms and the resulting static misallocation losses. Traditionally, these models have treated the evolution of a firm's productivity as exogenous and hence abstracted from the effects of financial frictions on firms' investments in productivity. Recent contributions endogenize firm productivity and show that this margin amplifies the output losses due to financial frictions (Ottonello and Winberry, 2024; Sui, 2024, 2025). Our paper differs from these contributions along two dimensions. First, we focus on the role of financial markets in shaping firms' innovation decisions, and quantify the aggregate effects of the resulting distortions, within a Schumpeterian endogenous growth model. Second, we augment the standard collateral-constraint environment with frictional equity markets, which allows us to study the relative role of equity versus debt finance in sustaining the scale-up of innovative firms.

Our work is closely related to a nascent literature that incorporates finance into endogenous growth models. Malamud and Zucchi (2019) build a Schumpeterian model in which innovative firms hoard cash to mitigate costly outside financing and show that financing fric-

tions slow down creative destruction by discouraging entry. Akcigit, Alp and Ercan (2021) and Geelen, Hajda and Morellec (2021) focus on the role of debt financing: the former shows that borrowing constraints select against high-growth firms, while the latter shows that debt fosters aggregate growth by stimulating entry. Akcigit, Dinlersoz, Greenwood and Penciakova (2022) and Greenwood, Han and Sánchez (2022) focus on venture capital and quantify its contribution to productivity growth in the U.S. Our contribution to this strand is threefold. First, we develop a unified framework that provides a comprehensive account of firms' financing structures, with internal funds, collateralized debt, and external equity. Second, we model external equity as the outcome of a costly search-and-bargaining process between firms and outside investors, which is novel relative to the literature. Third, equity's distinctive role operates entirely through selection: firms are indifferent to the source of their net worth, so equity's stronger association with innovation arises because investors direct funds to the firms with the highest return. This mechanism drives our quantitative results.

Our empirical evidence is related to the empirical literature on the financing of innovation; see Kerr and Nanda (2015) for a review. Relative to this literature, our contribution is threefold. First, we document that different sources of finance are allocated to systematically different firms: external-equity inflows are disproportionately received by firms that are smaller, younger, have lower net worth and collateral, and exhibit faster growth. Second, we characterize the dynamic relationship between innovation and capital injections, separately by source, documenting a strong, persistent and disproportionate correlation of innovation to external-equity injections relative to debt and internal funds. Third, we provide evidence consistent with a financial-frictions mechanism: external-equity inflows are followed by immediate and sustained increases in investment and sales, and the patenting correlation with equity injections is larger for small and young firms.

Outline The rest of the paper is structured as follows. Section 2 describes the data sources used in the paper and presents our main empirical findings. Section 3 introduces the theoretical framework and characterizes its Balanced Growth Path (BGP) equilibrium. Section 4 presents our calibration strategy and discusses some of the model results, showing that the model's predictions align well with the empirical evidence. Section 5 presents the results of our counterfactual exercises and policy experiments. Section 6 concludes.

2 Empirical Analysis

We use firm-level data from ORBIS to establish a set of empirical regularities that motivate our theory and that the structural model is later asked to reproduce. Our goal is to characterize the firm-level relationship between financial capital inflows and innovative activity and to document the patterns of selection and heterogeneity that discipline our model's mechanisms.

Three main facts organize the analysis. First, equity and debt fund different firms: external equity flows to smaller, younger, collateral-poor firms with lower net worth and faster growth. Second, firms' investment and innovation track financial capital inflows, and this co-movement is highly heterogeneous across financing sources – strong and disproportionate for external equity, and considerably weaker for debt and internal equity. Third, these patterns are consistent with imperfect access to external finance and with external equity targeting high-return firms.

2.1 Data Sources

We combine firm-level financial information from ORBIS Historical with patent data from ORBIS Intellectual Property. These datasets offer two main advantages: cross-country comparability through harmonized accounting and reporting standards, and coverage of both public and private firms for a nearly nationally representative dataset. We focus on the four largest continental European economies – France, Germany, Italy, and Spain – from 1995 to 2023.

Orbis Historical Firm-level balance sheet and income statement information comes from Orbis Historical, compiled by Moody's Bureau van Dijk (BvD). BvD collects firm-level data from various sources, such as national business registries, and harmonizes it into an internationally comparable format. The dataset provides balance sheets and income statements on an annual basis for both private and publicly traded companies.[1] Orbis Historical also reports information on firms' inputs, industry identifiers, and the year they were established. We describe our sample selection and data cleaning procedures in Appendix A.1 and provide key variable definitions in Appendix A.2.

[1] See Kalemli-Özcan, Sørensen, Villegas-Sanchez, Volosovych and Yeşiltaş (2024) for details on the dataset, as well as a discussion of the representativeness of the sample.

Orbis Intellectual Property We use patent data from Bureau van Dijk's ORBIS Intellectual Property (IP) database. ORBIS IP provides harmonized, firm-level information on patents and trademarks sourced from major international patent authorities. The database links intellectual property filings to standardized company identifiers from the ORBIS firm-level database, allowing patents and trademarks to be matched to firms' financial information and enabling the construction of firm-level measures of innovative activity.

Sample We restrict our analyses to the sample of *patenting* firms, composed of 170,432 firms that have applied for at least one (eventually granted) patent throughout their life. This sample corresponds to 1% of all firms across the four countries, for a total of 2,160,081 firm-year observations. Table A.1 in Appendix A.3 reports summary statistics for this sample of patenting firms.

2.2 Empirical Findings

Financial capital structure We begin by providing cross-sectional moments characterizing the role of debt and equity financing in firms' capital structure.

Table 1 provides a decomposition of a firm's financial capital into its three components: (i) financial debt; (ii) external equity, i.e. firm owners' external funds not generated inside the firm; (iii) internal equity, i.e. all the funds internally generated by the firm due to retained earnings or undistributed profits or losses. We also report descriptive statistics about the size and frequency of injections from each financial capital source, which we define as episodes in which the relevant financial capital component increases by more than 1%.

Internal equity is the largest component of financial capital (49.1%), followed by debt (34.3%), and external equity (16.6%). Through either debt or external equity, innovative firms raise 50.9% of their capital from external sources of finance. Among the two sources of external finance, 32.6% of external funds are raised from equity. Overall, equity (internal plus external) accounts for 65.7% of innovative firms' financial capital.

External-equity injections are infrequent: they occur in about 6.5% of firm-years, versus 35.7% for debt and 62.2% for internal equity. When they do occur, however, external-equity injections are sizeable: an injection adds on average 9.5% to the firm's existing financial capital, compared to 15.7% for debt issuances and 13.0% for internal-equity changes.

In Appendix A.3 we compare patenting and non-patenting firms in terms of their financial capital structure. Consistent with existing findings in the corporate finance literature, we document that: (i) patenting firms rely less on external sources of finance and more

Table 1: Financial capital structure.

	Mean	p10	p25	p50	p75	p90
Financial capital composition						
Debt share	0.343	0.014	0.132	0.340	0.496	0.668
External equity share	0.166	0.012	0.052	0.117	0.221	0.347
Internal equity share	0.491	0.120	0.330	0.494	0.677	0.849
Injection frequencies						
Debt	0.357					
External equity	0.065					
Internal equity	0.622					
Injection sizes (relative to financial capital)						
Debt	0.157	0.004	0.019	0.065	0.169	0.367
External equity	0.095	0.001	0.005	0.024	0.089	0.214
Internal equity	0.130	0.013	0.028	0.064	0.131	0.258

Notes: Variables defined as ratios are weighted by the denominator. Financial capital decomposition conditions on firms with positive financial capital and debt.

on internally generated funds; and (ii) among the different sources of external finance, patenting firms rely more on equity than debt finance.

Selection We characterize which firms raise each source of finance, to show that equity and debt reach systematically different firms. For each predetermined firm characteristic $x_{i,t-1}$, we regress it on the contemporaneous changes in total financial capital from each of the three financing sources, debt (Debt), external equity (EE) and internal equity (IE), entered jointly:[2]

$$x_{i,t-1} = \theta^E \, \Delta_0 \log(EE)_{it} + \theta^L \, \Delta_0 \log(\text{Debt})_{it} + \theta^I \, \Delta_0 \log(IE)_{it} + \gamma_{c(i)\,s(i)\,t} + u_{it}, \qquad (1)$$

where $\gamma_{c(i)\,s(i)\,t}$ are country×industry×year fixed effects. We estimate (1) for ten characteristics: net worth, tangible capital, total fixed capital, sales, employment, firm age, prior sales and employment growth, an indicator for patenting, and the intangible capital share.

Table 2 documents that external-equity inflows are systematically directed toward firms that, before the inflow, hold less net worth and are smaller, younger, and faster-growing. Column 1 shows that a one-log-unit increase in financial capital through external-equity inflow is directed toward firms that, before the inflow have about 0.76 SD lower sales,

[2] $\Delta_0 \log(X)_{it}$ with $X \in \{\text{EE}, \text{Debt}, \text{IE}\}$ represents the one-year log-difference in financial capital from source X while keeping the other two sources fixed: $\Delta_0 \log(X)_{it} = \log((X_{it} - X_{it-1}) + FC_{it-1}) - \log(FC_{it-1})$.

0.89 SD lower net worth, 0.56 SD lower tangible capital, 0.45 SD lower total fixed capital, and 0.39 SD lower employment; are roughly 0.45 SD younger; and have 0.45 SD higher prior sales growth and 0.24 SD higher prior employment growth. The same firms are also about 4.1 pp more likely to have applied for a patent in the previous year, and display a 8.4 pp higher intangible-capital share. Columns 2 and 3 show that this selection is an external-equity phenomenon, not a generic feature of raising funds: both differences are significant at the 1% level and, for most characteristics, as large as the external-equity coefficient itself. The implied internal-equity coefficients are essentially zero throughout, and the debt coefficients are an order of magnitude smaller than θ^E and, in the case of net worth, of the opposite sign.[3]

Table 2: Selection on predetermined firm characteristics by financing source.

Characteristic ($t-1$)	External equity	Ext. eq. − Debt	Ext. eq. − Internal eq.
Log net worth	-0.885^{***}	-0.997^{***}	-0.853^{***}
	(0.0162)	(0.0170)	(0.0162)
Log tangible capital	-0.559^{***}	-0.503^{***}	-0.562^{***}
	(0.0131)	(0.0131)	(0.0131)
Log total fixed capital	-0.446^{***}	-0.398^{***}	-0.447^{***}
	(0.0127)	(0.0127)	(0.0127)
Log sales	-0.759^{***}	-0.715^{***}	-0.766^{***}
	(0.0174)	(0.0174)	(0.0174)
Log employment	-0.392^{***}	-0.371^{***}	-0.395^{***}
	(0.0123)	(0.0124)	(0.0123)
Age	-0.452^{***}	-0.399^{***}	-0.453^{***}
	(0.0091)	(0.0092)	(0.0091)
Prior sales growth	0.450^{***}	0.356^{***}	0.444^{***}
	(0.0214)	(0.0215)	(0.0213)
Prior employment growth	0.242^{***}	0.162^{***}	0.241^{***}
	(0.0247)	(0.0248)	(0.0247)
Patenting	0.041^{***}	0.035^{***}	0.041^{***}
	(0.0037)	(0.0038)	(0.0037)
Intangible capital share	0.084^{***}	0.081^{***}	0.086^{***}
	(0.0039)	(0.0039)	(0.0039)

Notes: Each row regresses the predetermined ($t-1$) characteristic on the contemporaneous changes in financial capital from the denoted source $\Delta\log(\text{ext. equity})$, $\Delta\log(\text{debt})$, $\Delta\log(\text{int. equity})$ jointly. Col. 1: external-equity coefficient; cols. 2–3: its difference from the debt and internal-equity coefficients. The first eight characteristics are standardized to unit standard deviation; patenting and intangible capital share are in natural units. Country×industry×year FE; standard errors clustered by firm in parentheses. $^{*}p<0.10$, $^{**}p<0.05$, $^{***}p<0.01$.

[3]To isolate the extensive margin of selection, we re-estimate equation (1) replacing the continuous inflows with injection indicators. Results are in Table A.4 in Appendix A.3.

Dynamic correlations We estimate the dynamic response of innovation output to changes in financial capital using the local projections method of Jordà (2005), implemented in first differences.

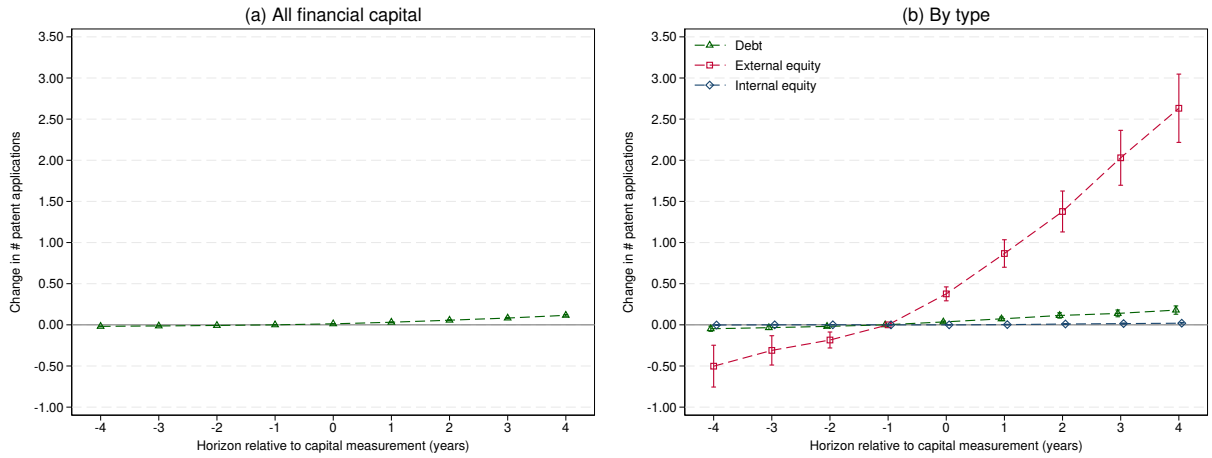
Let npat_{it} denote the number of patent applications filed by firm i in year t and let $\text{cum_npat}_{it} \equiv \sum_{\tau \leq t} \text{npat}_{i\tau}$. We regress the total number of patent applications filed between t and $t+h$, $\Delta_h \text{cum_npat}_{it+h} \equiv \text{cum_npat}_{it+h} - \text{cum_npat}_{it-1} = \sum_{s=t}^{t+h} \text{npat}_{is}$ on the contemporaneous change in financial capital and a set of predetermined controls:

$$\Delta_h \text{cum_npat}_{it+h} = \beta_h \Delta_0 \log(\text{FC})_{it} + \gamma X_{it-1} + \varepsilon_{it+h}, \tag{2}$$

where FC denotes total financial capital, X_{it-1} includes the stock of patents at $t-1$, cum_npat_{it-1}, lagged financial capital, $\log(\text{FC})_{it-1}$, lagged log sales, age fixed effects, and country×industry×year fixed effects to absorb country-sector-specific aggregate shocks. The coefficient β_h measures the average semi-elasticity of patenting between t and $t+h$ with respect to financial capital.

The left panel of Figure 1 reports the estimated semi-elasticities and their confidence intervals. An increase in financial capital is associated with a small but persistent rise in patenting activity. Quantitatively, a 10% increase in financial capital is associated with 0.01 additional patent applications over the four years following the shock. The response builds up steadily over time, consistent with the view that patenting activity is constrained by the availability of financial resources.

Figure 1: Financial capital and patenting.



Notes: This figure shows the dynamic relationship between financial capital and patenting activity for the pool of European patenting firms. The left panel plots the estimates from equation (2). The right panel presents the estimates for each type of financial capital in equation (3). Financial capital is the sum of debt, external equity and internal equity. Standard errors with 95% confidence intervals are clustered by firm.

To examine heterogeneity across funding sources, we extend the baseline specification by decomposing total financial capital into each of its three components:

$$\Delta_h \text{cum_npat}_{it+h} = \beta_h^L \Delta_0 \log(\text{Debt})_{it} + \beta_h^E \Delta_0 \log(\text{EE})_{it} + \beta_h^I \Delta_0 \log(\text{IE})_{it} + \gamma X_{it-1} + \varepsilon_{it+h}. \tag{3}$$

In this specification, X_{it-1} includes the lagged log levels of each financial capital component.

The right panel of Figure 1 shows that the effect of financial capital on patenting is highly heterogeneous across funding sources: capital injections in the form of external equity are more strongly correlated with an increase in patent applications. Specifically, a 10% increase in financial capital through external equity is associated with 0.26 additional patent applications over the four years after the injection. The effects of internal equity and debt are positive but significantly smaller in magnitude.

The estimated coefficients should only be interpreted as correlations rather than causal effects. Selection on unobservables is a natural driver of our results: among firms seeking funding, those with stronger growth prospects or more promising innovation opportunities may be more likely to attract external investors.Our proposed theory in Section 3 embraces selection as the mechanism giving rise to external equity's outsized correlation with patenting.

Financial frictions The mechanism proposed in this paper is that capital injections relax financial constraints that distort both production and innovation decisions. We test this channel by examining whether other firm-level outcomes that are closely related to production and innovative activity are also affected by changes in financial capital.[4]

In Appendix A.3, we show that both tangible and intangible investment and sales respond positively to increases in financial capital, particularly when the inflow takes the form of external equity. Using the same local projection framework, we find that both investment and sales increase on impact ($h = 0$) and steadily over subsequent years (Figures A.1 and A.2). These patterns are consistent with financially constrained firms using new funds to scale up production through capital deepening, generating an immediate growth in sales. Together with the evidence on the response of patenting activity, these results show that capital injections relax constraints along both margins that our theory will emphasize: the immediate response of investment and sales reflects the relaxation of the static production distortion, while the slow-moving response of patenting reflects the

[4]Ideally, one would study R&D expenditures directly. However, coverage of R&D spending in ORBIS is extremely limited.

relaxation of the dynamic R&D distortion.

To further probe the role of financial frictions, we show that the patenting response to external equity injections is concentrated among financially constrained firms. To do so, we re-estimate the local projection specification in equation (3) separately on two subsamples of firms that differ in observable proxies for the tightness of their financial constraints, and we compare the external-equity semi-elasticity across the two groups. We consider the following proxies for the shadow price of external finance: net worth, age, tangible capital, total physical capital (tangible + intangible), sales, and employment. For every proxy we sort firms into a "low" and a "high" group by splitting at the median of the predetermined $(t-1)$ value of the proxy, computed within industry$\times$country$\times$year cells; age is split at the conventional ten-year threshold.

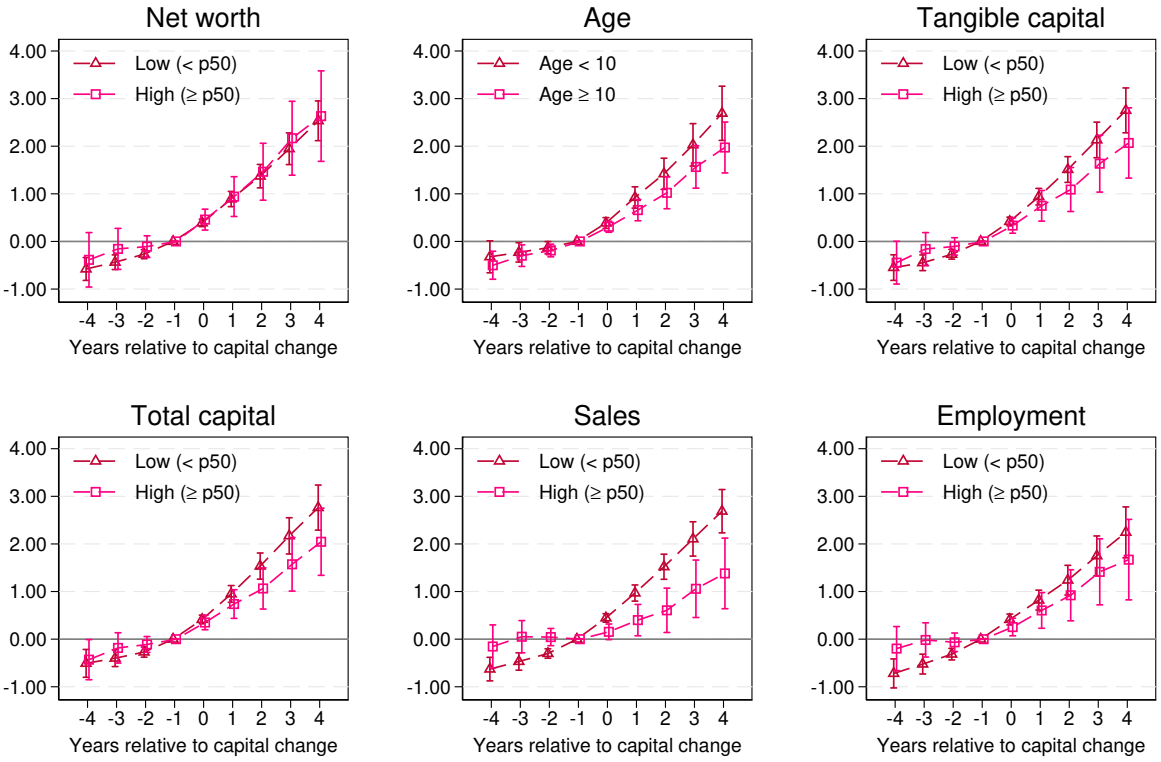
Figure 2 reports the results. In every panel, the response of patenting to an external-equity injection is systematically larger for the constrained group—low net worth, young, low tangible capital, low total capital, and low sales/employment—than for the unconstrained group, and the gap between the two opens up at longer horizons. The differentiation is most pronounced for age, tangible capital, total capital, sales, and employment, where by the fourth year the constrained group reaches roughly a 50% larger response than the unconstrained group; it is more muted for net worth, where the groups track each other closely. The direction of the gradient is nonetheless the same across all six proxies. For the other sources of financial capital, we observe the opposite pattern: where they differ at all, the responses of patenting to internal-equity and debt injections are smaller for the constrained group (see Figures A.3 and A.4).

The heterogeneity in the patenting responses to external-equity injections is what the model predicts and is central to our interpretation. The patenting response to external equity is amplified where financial constraints bind—among firms with a high marginal value of net worth, for whom relaxing the constraint most raises both the return to and the capacity to undertake R&D.

Taking stock Our empirical results show that external equity flows disproportionately toward firms that are smaller, younger, collateral-poor, faster-growing, and more intangible-intensive. Capital injections in the form of external equity are strongly associated with subsequent increases in firms' innovative activity, whereas debt and internal equity play a more limited role. Moreover, the patenting response to external equity injections is concentrated among financially constrained firms.

Taken together, these facts suggest that external equity displays the strongest correlation with innovation because it selects the firms whose innovation is most sensitive to financial

Figure 2: External equity and patenting, sample splits.



Notes: This figure shows the dynamic relationship between external equity and patenting activity for the pool of European patenting firms. Each panel plots the estimates from equation (3), estimated separately on two subsamples partitioned by the relevant proxy for financial constraints. Standard errors with 95% confidence intervals are clustered by firm.

capital. However, the extent to which these micro-elasticities translate into aggregate growth depends on the frequency, size, and distribution of capital injections across firms, and on how financing constraints interact with firm dynamics in equilibrium. To quantify these general-equilibrium mechanisms and assess their implications for long-run productivity growth, we turn to the structural model in the next section.

3 Model

We develop a Schumpeterian growth model of creative destruction with multi-product firms, physical capital in production, and frictional debt and equity markets. We use this framework to quantify the effects of financial frictions on aggregate innovation and long-run growth, and to study the mix of financial and innovation policies that best supports the scale-up of innovative firms.

3.1 Environment

Time is continuous, runs forever, and is indexed by t. The economy is populated by three types of agents: a representative household, a continuum of firms, and a representative financial intermediary. All agents are risk-neutral and infinitely-lived, and share a common time discount rate, $\rho > 0$. Firms may operate multiple product lines and produce using physical capital and labor. Firms finance their operations using internal funds, debt, and external equity. Firms can expand into new product lines through successful innovations, but may lose their products and eventually exit the economy due to creative destruction by competitors and entrants.

Preferences The representative household maximizes lifetime utility

$$U = \int_0^{+\infty} e^{-\rho t} \ln(C_t) dt, \tag{4}$$

where C_t is the consumption of a single final good, which is the numeraire and whose price we normalize to one. Every instant of time, the household inelastically supplies $L = 1$ units of labor at wage w_t. The household owns the production-sector firms and the financial intermediary, and accumulates assets A_t. The household's budget constraint satisfies $\dot{A}_t = r_t A_t + w_t - C_t$, where r_t is the rate of return on assets.

Production The final good, denoted Y_t, is produced competitively by a representative firm, combining intermediate goods Y_{jt} from a measure-one continuum of industries $j \in [0,1]$ using a Cobb-Douglas technology:

$$Y_t = \exp\left(\int_0^1 \ln(Y_{jt}) dj\right). \tag{5}$$

The cost-minimization problem of the representative final good producer generates the following unit-elastic demand for intermediate good j:

$$y_{jt} = \frac{Y_t}{p_{jt}}, \tag{6}$$

which implies that total sales in each product line j are $p_{jt} y_{jt} = Y_t$.

Industry j's output, Y_{jt}, is:

$$Y_{jt} = \sum_{f \in F_{jt}} y_{fjt}, \tag{7}$$

where y_{fjt} is the quantity of good j bought from firm f and F_{jt} denotes the set of firms competing in industry j at time t. Thus, each product can potentially be produced by a large number of firms. Different products j and j' are imperfect substitutes, whereas there is perfect substitutability between different firms within a product.

Firms can produce multiple products. A firm f can produce product j using physical capital, k_{fjt}, and labor, l_{fjt}, according to the constant-returns-to-scale technology:

$$y_{fjt} = k_{fjt}^{\alpha}(q_{fjt}l_{fjt})^{1-\alpha}, \tag{8}$$

where q_{fjt} is the labor-augmenting productivity of the firm, and $\alpha \in (0,1)$ is the output elasticity of capital.[5] Physical capital depreciates at rate δ. We denote by $\mathcal{M}_{ft}$ the set of product lines operated by firm f at time t and by $k_{ft} = \sum_{j\in\mathcal{M}_{ft}} k_{fjt}$ and $l_{ft} = \sum_{j\in\mathcal{M}_{ft}} l_{fjt}$ the total capital and labor it employs across these product lines.[6]

Market structure At every instant, firms engage in Bertrand competition on product markets. Given perfect substitutability among the varieties produced by different firms, only the cheapest variety will be sold in equilibrium. The firm with the highest productivity, which we denote by $q_{jt} \equiv \max\{q_{fjt}\}$, can produce good j at the highest price that leaves its competitors with non-positive profits. However, as it will become apparent later, in our setting, due to the presence of financial frictions, this firm might not be able to satisfy the entire demand for the good at the equilibrium price. In this case, we assume that any residual demand will be met by the pool of competitors, who share the same productivity $q_{jt}^{F} < q_{jt}$.

Innovation Firms grow by expanding into new markets at the expense of other producers. Specifically, an incumbent can generate innovations which improve the productivity of products currently produced by others. To add new product lines to its portfolio, a firm invests in R&D.

An incumbent firm that currently produces $m \in \mathbb{N}^{+}$ products and invests c_t^z units of the final good in R&D generates a Poisson arrival rate Z of innovation according to the production function: $Z = (\frac{1}{\xi}\frac{c_t^z}{Y_t})^{\frac{1}{\psi}} m^{\frac{\psi-\sigma}{\psi}}$, where $\psi > 1$, $\sigma > 0$ is the elasticity of R&D cost with respect to firm size, and $\xi \in \{\xi_L, \xi_H\}$ is a firm's innovation productivity type, with $\xi_L > \xi_H > 0$.[7] Transitions across innovation productivity types are exogenous. Low-type

[5] We need productivity to be labor-augmenting to ensure the existence of a Balanced Growth Path (BGP) equilibrium.

[6] In what follows, we drop the subscript f for ease of exposition when this does not cause confusion.

[7] This functional form allows for an arbitrary amount of scaling of R&D with firm size.

firms become high-type at some arrival rate $\iota_{L\to H}$, and the reverse occurs with arrival rate $\iota_{H\to L}$. This production function implies that the firm's cost function to achieve an innovation intensity of $z \equiv Z/m$ is:

$$c_t^z(z,m) = \xi z^{\psi} m^{\sigma} Y_t. \tag{9}$$

Innovations apply to a randomly chosen product in the interval $[0,1]$, and enable the incumbent to improve its leading productivity by a factor $\lambda > 1$. The innovator then becomes the new producer of this improved product, displacing the previous producer. The stochastic process for innovation leads to heterogeneity in the size of firms.

On the other hand, an incumbent faces the possibility that some other firm innovates on a good it is currently producing. In this case, the incumbent producer loses the good from its portfolio. This event occurs at the endogenous rate of creative destruction τ_t, which is taken as given by each individual firm and is determined in equilibrium by the aggregate rate of innovation. When an incumbent loses its last remaining product, it exits the economy.

Dividend payouts Firms distribute dividends to their shareholders. The present value of all future dividend payouts determines the firm's value to its shareholders. As in Jermann and Quadrini (2012), dividend payouts are subject to quadratic adjustment costs that create a wedge between the value of the payout to the shareholder and the cost to the firm. Letting $d \geq 0$ be the firm's payout intensity (total dividend payout over firm size), the cost to the firm of the dividend payout d is $c_t^d(d) = (d + \frac{\kappa}{2}d^2)mY_t$ in units of the final good, where $\kappa > 0$ is the parameter governing the scale of the adjustment costs.

Finance To finance their expenditures, firms can rely on internal funds and costly external financing, by issuing one-period risk-free debt or equity claims to future dividends.

Let b_t denote the amount of borrowing of the firm. The budget constraint of an intermediate good producer is then:

$$r_t b_t + \dot{k}_t + \delta k_t + w_t l_t + c_t^d(d) + c_t^z(z) = \sum_{j\in\mathcal{M}_t} p_{jt} y_{jt} + \dot{b}_t.$$

Defining the firm's net worth, or equity, as $N_t := k_t - b_t$, we can rewrite this constraint as:

$$\dot{N}_t = \Pi_t + r_t N_t - c_t^z(z,m) - c_t^d(d) \tag{10}$$

where $\Pi_t \equiv \sum_{j\in\mathcal{M}_t} p_{jt} y_{jt} - w_t l_t - R_t k_t$ are production profits, with $R_t = r_t + \delta$. Equation (10) describes the continuous and deterministic changes in a firm's net worth. The drift of

net worth is given by profits from production plus the rent flow from equity, net of R&D expenditures and dividend payouts, inclusive of the adjustment costs. At every point in time, firms face a borrowing limit $N_t \geq \underline{N}$, where $\underline{N} \in (-\infty, 0)$.

We assume that the amount the producer can borrow is limited by a collateral constraint that requires that its debt does not exceed a fraction of its capital stock plus a fraction of revenues (as in Drechsel (2023), for instance):

$$b_t \leq \vartheta_k k_t + \vartheta_y \sum_{j \in \mathcal{M}_t} p_{jt} y_{jt}. \tag{11}$$

The parameters $\vartheta_k \leq 1$ and $\vartheta_y \leq 1$ capture the extent of assets and revenue pledgeability in external borrowing, respectively. Given the definition of net worth, the collateral constraint can also be written as:

$$k_t \leq \theta_n N_t + \theta_y \sum_{j \in \mathcal{M}_t} p_{jt} y_{jt}, \tag{12}$$

where $\theta_n := (1 - \vartheta_k)^{-1}$ and $\theta_y := \vartheta_y (1 - \vartheta_k)^{-1}$.

The net worth of a firm can jump discontinuously in case of an equity injection from outside investors, which represents the second source of external finance in our model. A meeting between investors and an intermediate-good firm occurs at the exogenous arrival rate χ. When a meeting occurs, outsiders may make an equity injection in exchange for acquiring a share of the firm, as described in detail below. We assume that the sale of equity stakes to outsiders involves a cost that is increasing and convex in the size of the equity injection.

Financial sector The financial sector is populated by a representative firm, owned by the household, which acquires equity stakes of firms in the production sector. Denoting by $(1 - \omega_{ft})$ the share of firm f, with value V_{ft}, owned by the representative financial firm at time t, the value from all equity stakes is: $I_t = \int_{\mathcal{F}} (1 - \omega_{ft}) V_{ft} df$, with $\mathcal{F}$ denoting the set of production sector firms. The intermediary is risk-neutral and can save any funds not injected at the return r_t, so its marginal value of a unit of funds is one.

When there is a meeting between the financial intermediary and a production-sector firm, the financial intermediary may acquire an equity stake in the firm. This occurs if the transaction generates a surplus, that is, if the joint value of both firms after trade is greater than the sum of their outside options. The financier then transfers $T_{jt}(\omega)$ units of the final good to the firm in exchange for an ownership share $(1 - \omega) \in (0, 1]$. The resulting transaction surplus is split between the two parties following a Nash bargaining protocol, where the firm's bargaining weight is $\eta \in (0, 1)$.

Entry At any point in time, a mass of potential entrants of measure one invest in innovation to displace incumbents and enter a market. In order to generate a Poisson arrival rate e of successful firm creation, a potential entrant must incur an R&D cost flow in terms of the final good given by $c_t^e(e) = \tilde{\xi} e^{\psi} Y_t$, with $\tilde{\xi} > 0, \psi > 1$. Upon successful entry, the firm starts operating with an initial level of net worth drawn from an exponential distribution with parameter v. A fraction $\beta \in [0,1]$ of entrants are low innovation productivity type and a fraction $1-\beta$ are high type. We assume that β equals the share of low types from the stationary distribution of the continuous-time Markov chain defined by the flow rates $\iota_{H\to L}$ and $\iota_{L\to H}$. That is, $\beta = \frac{\iota_{H\to L}}{\iota_{H\to L}+\iota_{L\to H}}$.

3.2 Equilibrium

Throughout, we consider a Balanced Growth Path (BGP) equilibrium in which aggregate output, capital and consumption grow at a constant rate $g > 0$.

3.2.1 Household decisions, and firm's static equilibrium conditions

The household's problem leads to the standard Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho, \tag{13}$$

which determines the growth rate of consumption. Since consumption growth is constant, it follows that $r_t = r$.

Because of constant returns to scale in production, Bertrand competition, and perfect substitutability between different varieties within a product, for any product j, only the firm with the highest productivity produces in equilibrium, provided it is unconstrained. If constrained, however, the leader in product line j might not be able to satisfy the entire demand for the product. We deal first with the case of an unconstrained leader and then move to the case in which the borrowing constraint is binding.

If the leader is unconstrained, it will be the only producer of good j in equilibrium. However, the presence of competing producers constrains the leading firm to engage in limit pricing: the leader charges the highest price that makes the fringe of followers inactive:

$$p_{jt} = \frac{1}{{q_{jt}^F}^{1-\alpha}} \underbrace{\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}}_{:=h_t} = \mu \frac{h_t}{{q_{jt}}^{1-\alpha}}, \text{ with } \mu \equiv \left(\frac{q_{jt}}{q_{jt}^F}\right)^{1-\alpha} = \lambda^{1-\alpha}, \tag{14}$$

where μ is the markup. Given the pricing schedule (14) and demand (6), the static profit maximization problem of the leader delivers:

$$k_{jt}^* = \frac{\alpha}{R_t}\frac{Y_t}{\mu} \quad \text{and} \quad l_{jt}^* = \frac{1-\alpha}{w_t}\frac{Y_t}{\mu}.$$

These optimality conditions are identical for any j, which implies $k_{jt} = k_t/m_t$ and $l_{jt} = l_t/m_t$: the total capital and labor used by the firm are split equally across all product lines. It follows that an unconstrained producer obtains the following flow of profit from good j:

$$\Pi_{jt} = \left(p_{jt} - \frac{h_t}{q_{jt}^{1-\alpha}}\right) y_{jt} = \left(1-\mu^{-1}\right) Y_t \equiv \Pi_t, \tag{15}$$

so that each good generates the same profit flow and an unconstrained firm operating m_t product lines earns profits $m_t\Pi_t$.

If $k_{jt}^* > \frac{1}{m_t}(\theta_n N_t + \theta_y \sum_{j\in\mathcal{M}_t} p_{jt}y_{jt})$, the borrowing constraint binds and the unconstrained solution is unattainable. The leading firm might be unable to satisfy the entire demand at price $p_{jt} = \mu h_t/q_{jt}^{1-\alpha}$, but it is still constrained in price setting by the presence of competing producers. In particular, we can show that it is still optimal for the leader to set its price according to equation (14).[8] In this case, the leader levers up to the maximum extent possible, setting

$$k_{jt} = \frac{1}{m_t}(\theta_n N_t + \theta_y \sum_{j\in\mathcal{M}_t} p_{jt}y_{jt}) \equiv \bar{k}_t(N_t, m_t). \tag{16}$$

Given the price schedule and the choice of capital, the profit-maximizing labor input of a constrained leader is

$$l_{jt}^C = \frac{1-\alpha}{\alpha}\frac{R_t}{w_t}\mu^{\frac{1}{\alpha}}\bar{k}_t(N_t, m_t),$$

while the labor input that satisfies the entire demand for product j is given by:

$$l_{jt}^D = \frac{1-\alpha}{w_t}\left[\left(\frac{\alpha}{R_t\bar{k}_t(N_t,m_t)}\right)^{\alpha}\frac{Y_t}{\mu}\right]^{\frac{1}{1-\alpha}},$$

so that a constrained leader employs

$$l_{jt} = \min\left\{l_{jt}^C, l_{jt}^D\right\}.$$

[8] We prove this result formally in Appendix B.1.

Comparing the two expressions, $l_{jt}^C \leq l_{jt}^D$ if and only if $\bar{k}_t(N_t, m_t) \leq \underline{k}_t \equiv \alpha Y_t/(R_t \mu^{1/\alpha})$. Whenever $l_{jt} < l_{jt}^D$, we assume that the residual demand left by the leader is met by the followers, from which they obtain zero profits given the equilibrium price. Note that, as in the unconstrained case, the firm devotes the same amount of inputs to production in each line.

Static profits from operating any product line for a constrained leader are therefore:

$$\Pi_t(N_t, m_t) = \begin{cases} \bar{k}_t(N_t, m_t) R_t \left(\mu^{\frac{1}{\alpha}} - 1\right) & \text{if} \quad \bar{k}_t(N_t, m_t) \leq \underline{k}_t, \\ Y_t - \bar{k}_t(N_t, m_t) R_t - (1-\alpha)\left(\left(\frac{\alpha}{\bar{k}_t(N_t, m_t) R_t}\right)^{\alpha} \frac{Y_t}{\mu}\right)^{\frac{1}{1-\alpha}} & \text{if} \quad \bar{k}_t(N_t, m_t) > \underline{k}_t, \end{cases} \tag{17}$$

and, since each good generates the same profit flow, total firm profits are $m_t \Pi_t(N_t, m_t)$.

The definition of $\bar{k}_t(N_t, m_t)$ in (16) is implicit, because the collateral base includes the firm's own sales, which in turn depend on the capital the firm is able to finance. Making it explicit is instructive, because the two cases in (17) behave very differently. Consider first a leader that serves the entire demand for its products. Revenue in each line is then $p_{jt} y_{jt} = Y_t$, so that $\sum_{j \in \mathcal{M}_t} p_{jt} y_{jt} = m_t Y_t$ and

$$\bar{k}_t(N_t, m_t) = \theta_n \frac{N_t}{m_t} + \theta_y Y_t.$$

An additional product line contributes its own sales to the collateral base, but absorbs the capital requirement of a full line. Capital per line therefore falls with m_t, and total profits rise with the number of product lines less than proportionally.

Consider next a leader that cannot serve the entire demand for its products. Substituting l_{jt}^C into the production function and using the price schedule (14), revenue in each line is proportional to the capital employed in that line,

$$p_{jt} y_{jt} = \frac{\mu^{\frac{1}{\alpha}} R_t}{\alpha} \bar{k}_t(N_t, m_t) \equiv \Gamma_t \bar{k}_t(N_t, m_t). \tag{18}$$

Substituting into (16) gives

$$\bar{k}_t(N_t, m_t) = \frac{\theta_n N_t}{m_t \left(1 - \theta_y \Gamma_t\right)},$$

so that the firm's total capital stock

$$m_t \bar{k}_t(N_t, m_t) = \frac{\theta_n N_t}{1 - \theta_y \Gamma_t},$$

depends on net worth alone.[9] Total profits are therefore

$$m_t \Pi_t(N_t, m_t) = \frac{\theta_n R_t \left(\mu^{\frac{1}{\alpha}} - 1\right)}{1 - \theta_y \Gamma_t} N_t, \tag{19}$$

which is linear in net worth and independent of the number of product lines. Equivalently, profits per line fall at rate $1/m_t$, and the marginal product line is worth nothing to the firm. The borrowing constraint therefore does more than lower the level of profits: where it binds tightly enough that demand goes unserved, it removes the return to operating an additional product line altogether.

Figure 3 illustrates these results. The left panel partitions the state space into three regions. In the lightest region the collateral constraint is slack and the firm operates every product line at the unconstrained scale. In the intermediate region the constraint binds, but the firm still serves the entire demand for its products, which it does at a capital intensity below the unconstrained optimum. In the darkest region the constraint binds so tightly that the firm produces less than the quantity its customers would absorb.[10] For each given number of product lines, m, there exists a threshold level of net worth below which the firm is financially constrained, and a second, lower threshold below which it can no longer serve demand. As the number of product lines operated by the firm increases, the net worth required to operate every line at the unconstrained scale rises proportionally, and hence both thresholds increase. If net worth accumulation does not keep pace with growth in the product space, the firm crosses these boundaries and finds itself financially constrained.

[9] The fixed point is well defined provided $\theta_y \Gamma_t < 1$. Each unit of capital generates Γ_t units of sales, which relax the constraint by $\theta_y \Gamma_t$ units of capital; if this exceeded one, the collateral base would be self-sustaining and the constraint would never bind.

[10] Comparing $\bar{k}_t(N_t, m_t) = \theta_n N_t / m_t + \theta_y Y_t$ with k^*_{jt} and with $\underline{k}_t$ delivers the two boundaries in closed form: the firm is unconstrained if and only if

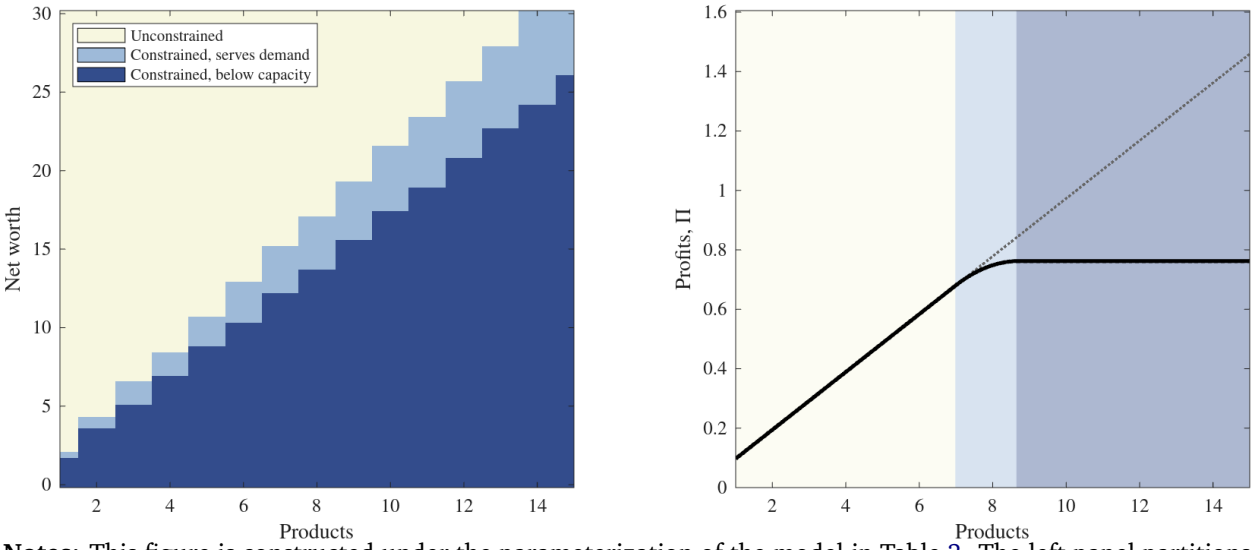
$$\frac{N_t}{m_t} \geq \frac{Y_t}{\theta_n} \left(\frac{\alpha}{R_t \mu} - \theta_y \right),$$

and it serves the entire demand for its products if and only if

$$\frac{N_t}{m_t} \geq \frac{Y_t}{\theta_n} \left(\frac{\alpha}{R_t \mu^{1/\alpha}} - \theta_y \right).$$

The right panel traces the consequences for profits. Holding net worth fixed and increasing the number of product lines moves the firm rightward across the left panel, from the unconstrained region into the two constrained ones. While the firm is unconstrained, profits increase linearly in the number of product lines and coincide with the dotted line, which plots profits in the absence of the collateral constraint. Once the constraint begins to bind, profits continue to increase but at a declining rate, so that the return from expansion into new product lines is smaller than in the unconstrained region. Once the firm can no longer serve the demand for its products, profits become exactly flat. This collapse in the marginal return to product lines distorts financially constrained firms' dynamic innovation decisions, which we discuss next.

Figure 3: Debt frictions and firms' static profits.



Notes: This figure is constructed under the parameterization of the model in Table 3. The left panel partitions the state space according to whether the collateral constraint is slack, binding with the firm still able to serve the demand for its products, or binding so tightly that output falls short of demand. The right panel plots static profits against the number of product lines, holding net worth fixed at its average; the shaded regions correspond to those of the left panel. The dotted line plots profits in the absence of the collateral constraint.

3.2.2 Dynamic problem of incumbent firms

Given the solution to the static problem, the incumbent firm must choose its dividend payout, $d \geq 0$, and innovation, z, policies to maximize shareholder value. The value of an incumbent firm depends on its net worth/equity stock, N_t, the number of product lines it owns, $m \geq 1$, and its innovation productivity, ξ. As we prove in Appendix B.2, the value function is proportional to aggregate output

$$V_t(N_t, m, \xi) = v(n_t, m, \xi) Y_t, \tag{20}$$

where $n_t = N_t/Y_t$ and the normalized value function solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\rho v(n_t,m,\xi) = \max_{d\geq 0,z} \Big\{ & md + \frac{\partial v(n_t,m,\xi)}{\partial n_t}\left(\pi(n_t,m) + \rho n_t - \xi z^{\psi} m^{\sigma} - \left(d + \frac{\kappa}{2}d^2\right) m\right) \\ & + zm(v(n_t,m+1,\xi) - v(n_t,m,\xi)) \\ & + \sum_{\xi' \in \{\xi_L,\xi_H\}} \iota_{\xi\to\xi'}\left(v(n_t,m,\xi') - v(n_t,m,\xi)\right) \\ & + \chi v^s(n_t,m,\xi) + \tau m(v(n_t,m-1,\xi) - v(n_t,m,\xi))\Big\} \qquad (21) \\ \text{s.t.} \quad & n_t \geq \underline{n},\end{aligned}$$

for $m \geq 1$, where $\pi(n_t,m) = m\Pi_t(N_t,m)/Y_t$, $\underline{n} = \underline{N}_t/Y_t$ and τ is the aggregate rate of creative destruction. We treat a firm in state $m = 0$ as having permanently exited. In this event, residual firm assets are distributed to shareholders as dividends. The exit value of a firm with net worth n_t is therefore $v(n_t,0,\xi) = d_0(n_t)$, where $d_0(n_t)$ solves $\frac{\kappa}{2}d_0^2 + d_0 = n_t$.

The HJB equation (21) shows how the discounted (normalized) value of an incumbent firm changes over time.

The first term is current-period dividends, while the other terms account for changes in firm value over time. The second term captures changes in firm value due to changes in net worth. $\partial v(n_t,m)/\partial n_t$ is the shadow value of funds, i.e. the marginal value of keeping a unit of resources inside the firm. This object represents the opportunity cost of spending resources outside the firm, and is heterogeneous across firms, depending on their financial positions. In particular, the returns to resource accumulation are largest for firms with little equity per product line, for which the collateral constraint is binding or close to be binding. The term in the second line accounts for the change in firm value conditional on adding one product line to its portfolio (i.e. increasing in size to $m+1$) as a result of successful innovation. The third line captures the change in value due to transitioning from one innovation productivity type to the other. The first term in the fourth line represents the value from matching with outside investors, which we derive below. Finally, as captured by the last term, a firm with $m \geq 1$ products faces a Poisson hazard τm of becoming a firm of size $m-1$.[11]

The first order conditions to problem (21) determine the optimal dividend payout and

[11]The firm's dynamic problem is subject to a state constraint on equity, $n_t \geq \underline{n}$. The presence of this constraint does not affect first order conditions, but simply requires that the value of the firm satisfies a boundary condition. Specifically, it requires that, when at the boundary, the drift of net worth cannot be negative.

innovation policies of the firm:

$$\frac{\partial v(n_t, m, \xi)}{\partial n_t}(1 + \kappa d^*) = 1 + \Xi, \tag{22}$$

$$\frac{\partial v(n_t, m, \xi)}{\partial n_t}\psi \xi z^{*\psi-1} m^{\sigma-1} = v(n_t, m+1, \xi) - v(n_t, m, \xi), \tag{23}$$

where Ξ is the Lagrange multiplier on the non-negativity constraint on dividends.

Equations (22)-(23) display the consequences of financial considerations for firms' expenditure decisions. The left-hand sides of these equations represent the marginal costs of each activity. On top of the pecuniary costs, these marginal costs also include the shadow value of funds. Indeed, expending resources, instead of accumulating them, reduces firm's equity, making the borrowing constraint more likely to bind, and hence reducing firm value. The presence of financial frictions, by increasing $\partial v(n_t, m)/\partial n_t$, has therefore the effect of increasing firms' marginal costs, especially for highly constrained firms. In particular, financially constrained firms face a higher marginal cost of R&D expenditures. On the marginal benefit side, firms with different amounts of equity per product line face a different marginal benefit from expanding into a new product line. Thus, heterogeneity in n_t/m generates dispersion in returns to R&D for fixed m. In particular, firms with low net worth per product line have low incentives to innovate, as they do not have enough equity to rent a sufficient amount of capital for production. This has the effect of reducing the marginal benefit from innovation for highly financially constrained firms. Thus, the innovation incentives of firms with low equity per product line are lower for two reasons: their R&D costs are higher and their benefits from expansion are lower.

3.2.3 Outside equity

We now derive the change in incumbent value due to selling shares to outside investors, $v^s(\cdot)$. In case of a meeting, if the financial intermediary acquires a share $1-\omega$ of a firm in state (n_t, m, ξ) and makes an equity injection T_t, its change in value is given by $(1-\omega)v(n_t + T_t, m, \xi)$ net of the transaction's costs, where $v(n_t + T_t, m, \xi)$ is the post-trade value of the production-sector firm. In particular, we assume the transfer of resources from the fund to the firm is subject to frictions, such that to increase firm's equity by T_t, the fund incurs $c_f(T_t) = T_t + \frac{\zeta}{2}T_t^2$ in costs, with $\zeta > 0$.[12] For the owner of the firm, the value of the firm after trade will be $\omega v(n_t + T_t, m, \xi)$. The surplus from issuing outside equity is

[12]We think of $c_f(T_t)$ as a reduced-form way of capturing issuance costs or investor skepticism about issuing large amounts of equity.

therefore given by:

$$\Omega_t(n_t, m, \xi, T_t) = v(n_t + T_t, m, \xi) - v(n_t, m, \xi) - c_f(T_t), \tag{24}$$

where $v(n_t, m, \xi)$ is the pre-transaction value of the firm, which is the value of problem (21). A trade will occur if and only if this surplus is positive. This will be the case whenever: $v(n_t + T_t, m, \xi) - v(n_t, m, \xi) > c_f(T_t)$, i.e. when the increase in firm value is larger than the transfer cost paid by the financial intermediary, inclusive of all costs. In this case, the financier's funds are worth more in the hands of the firm, and the equity sale allows this transfer to take place.

Upon meeting, the investors and the firm negotiate the transaction that maximizes their joint surplus. The surplus-maximizing transaction price solves:

$$T_t^*(n_t, m, \xi) = \arg\max_{T_t \geq 0} \Omega_t(n_t, m, \xi, T_t), \tag{25}$$

and is pinned down by the following first-order condition:

$$\frac{\partial v(n_t + T_t^*, m, \xi)}{\partial n_t} = 1 + \zeta T_t^*. \tag{26}$$

The left-hand side is the firm's shadow value of funds after the transfer; the right-hand side is the intermediary's marginal cost of providing the funds. In order to determine the share sold to the fund, we assume that the two parties bargain over total surplus following a Nash bargaining protocol, where the firm's bargaining weight is $\eta \in (0, 1)$. Under this assumption, we can find ω^* (and hence the share sold to the fund, $1 - \omega^*$) by solving:

$$\omega^* v(n_t + T_t^*, m, \xi) = v(n_t, m, \xi) + \eta \underbrace{\left(v(n_t + T_t^*, m, \xi) - v(n_t, m, \xi) - c_f(T_t^*)\right)}_{:=\Omega^*(n_t, m, \xi)},$$

which implies

$$\omega^* = (1 - \eta)\left(\frac{v(n_t, m, \xi)}{v(n_t + T_t^*, m, \xi)}\right) + \eta\left(1 - \frac{c_f(T_t^*)}{v(n_t + T_t^*, m, \xi)}\right). \tag{27}$$

As $\eta \to 0$, the share sold to the fund fully compensates for the gains from the increase in resources, and the firm is indifferent about the transaction taking place, $\omega^* v(n_t + T_t^*, m, \xi) = v(n_t, m, \xi)$. By contrast, when $\eta \to 1$, ω^* is large enough to make the fund just as well off as before the transaction, $(1 - \omega^*)v(n_t + T_t^*, m, \xi) = c_f(T_t^*)$.

Given T_t^* and ω^*, the expected change in firm value from a meeting with outside

investors is given by:

$$v^s(n_t, m, \xi) = \omega^* v(n_t + T_t^*, m, \xi; \eta) - v(n_t, m, \xi) = \eta \Omega^*(n_t, m, \xi). \tag{28}$$

3.2.4 Firm creation

Next, we determine the equilibrium entry rate. A mass of potential entrants of measure one invest in R&D to capture a random product line, displacing the incumbent. In case of successful innovation, a new firm is created, with (normalized) value $v(n_t^0, 1, \xi)$. Net worth at entry is drawn from an exponential distribution with parameter v, $n_t^0 \sim \exp(v)$. Each entrant has probability $\beta \in [0, 1]$ of entering as a low innovation productivity type and probability $1 - \beta$ of entering as a high type. The equilibrium arrival rate of a potential entrant's innovation is therefore determined as the solution to the following problem:

$$\max_e \Big\{ -c_t^e(e) + e\Big(\mathbb{E}_{n_t^0}[\beta v_t(n_t^0, 1, L) + (1-\beta) v_t(n_t^0, 1, H)]\Big)\Big\}, \tag{29}$$

where the first term in the maximization is the R&D cost needed to found a new firm and the second is the expected flow value from founding one. The equilibrium arrival rate of a potential entrant's innovation is therefore given by:

$$e^* = \left(\frac{\mathbb{E}_{n_t^0}[v_t(n_t^0, 1, H) - \beta(v_t(n_t^0, 1, H) - v_t(n_t^0, 1, L))]}{\psi \tilde{\xi}} \right)^{\frac{1}{\psi - 1}}. \tag{30}$$

To the extent that n_t^0 is sufficiently low, the entrepreneur will be financially constrained upon successful entry, and its profits will be low until the firm builds up sufficient equity. Through this channel, financial frictions impact new business entry.

3.2.5 Closing the model

To close the equilibrium characterization, we impose market clearing conditions. In the general case, the demand for product j will be jointly satisfied by the leader and the pool of followers. Let i denote the leader and $-i$ denote the pool of followers in product line j. Then, total sales in industry j are:[13]

$$Y_t = p_{ijt} y_{ijt} + p_{-ijt} y_{-ijt} = p_{ijt}(y_{ijt} + y_{-ijt}) = h_t(\mu k_{ijt}^{\alpha} l_{ijt}^{1-\alpha} + k_{-ijt}^{\alpha} l_{-ijt}^{1-\alpha}).$$

[13] In the special case in which the leader firm is able to satisfy the entire demand for product j, $y_{-ijt} = 0$.

This implies

$$k_{ijt}^{\alpha} l_{ijt}^{1-\alpha} + (\mu)^{-1} k_{-ijt}^{\alpha} l_{-ijt}^{1-\alpha} = \frac{Y_t}{h_t} \mu^{-1},$$

or equivalently:

$$X_t^{\alpha} = \frac{Y_t}{h_t} \mu^{-1}, \tag{31}$$

where $X_t := \left[\int_0^1 \left(k_{ijt}^{\alpha} l_{ijt}^{1-\alpha} + (\mu)^{-1} k_{-ijt}^{\alpha} l_{-ijt}^{1-\alpha} \right) dj \right]^{\frac{1}{\alpha}}$.

Using the final good aggregator and substituting firms' technology and input choices, we find:

$$\begin{aligned} Y_t &= \exp\left(\int_0^1 \ln y_{jt} dj \right) = \exp\left(\int_0^1 \ln(y_{ijt} + y_{-ijt}) dj \right) \\ &= \underbrace{\exp\left(\int_0^1 (1-\alpha) \ln q_{jt} dj \right)}_{:= Q_t^{1-\alpha}} \exp\left(\int_0^1 \ln(k_{ijt}^{\alpha} l_{ijt}^{1-\alpha} + (\mu)^{-1} k_{-ijt}^{\alpha} l_{-ijt}^{1-\alpha}) dj \right) \\ &= Q_t^{1-\alpha} \exp\left(\int_0^1 \ln \frac{Y_t}{h_t} \mu^{-1} dj \right), \end{aligned}$$

which implies

$$h_t = Q_t^{1-\alpha} / \mu. \tag{32}$$

Here, Q_t is the aggregate productivity index, which is defined as the average productivity of the leading firm across product lines. Combining (31) and (32), we can write aggregate output as:

$$Y_t = X_t^{\alpha} Q_t^{1-\alpha}. \tag{33}$$

Final output is used for household consumption, investment, and to pay for R&D costs and entry costs.[14] As we show in Appendix B.5, the aggregate resource constraint reads:

$$1 = \frac{C_t}{Y_t} + (g+\delta) \frac{K_t}{Y_t} + \int_0^1 \xi z_{jt}^{\psi} m_{jt}^{\sigma-1} dj + \tilde{\xi} e_t^{\psi} + \frac{\Sigma_t}{Y_t} \tag{34}$$

where: $\Sigma_t = \left(\int_0^1 \frac{\kappa}{2} d_{jt}^2 dj + \chi \int \frac{\zeta}{2} T_t^2 d\phi_t \right) Y_t$ are dividend payout and transaction costs paid by firms and the financial intermediary; K_t is the aggregate capital stock; z_{jt}, d_{jt} and m_{jt} denote the policies and size of the firm that owns product line j, so that $\xi z_{jt}^{\psi} m_{jt}^{\sigma-1} Y_t$ is that firm's R&D cost per line, and $\int \cdot d\phi_t$ integrates over the distribution of firms defined below.[15]

[14] Recall that dividend payout costs are a resource drain in this economy.

[15] Because these terms are a reduced way of capturing frictions, we treat them as resource drains.

Finally, we characterize the stationary distribution of firms in BGP. Let $\phi(n,m,\xi,t)$ be the density of the joint distribution of n, m and ξ at time t and $\Phi(n,m,\xi,t)$ be the corresponding CDF in the net worth dimension: $\Phi(n,m,\xi,t) = \Pr(n_t \leq n, m_t = m, \xi_t = \xi)$. Moreover, let $s(n,m,\xi,t)$ denote the optimal "saving" policy function implied by optimal payout and innovation decisions, and $\varrho(n)$ the probability density function over net worth at entry. As we show in Appendix B.3, in a stationary equilibrium $\phi(n,m,\xi)$ is determined by the Kolmogorov Forward (KF) equation:

$$\begin{aligned} 0 =& \frac{\partial}{\partial n}\left[-s(n,m,\xi)\phi(n,m,\xi) - \chi\int_{\mathcal{S}_t(n,m,\xi)} d\Phi(\nu,m,\xi)\right] \\ &- (mz(n,m,\xi) + \iota_{\xi\to\xi'} + m\tau)\phi(n,m,\xi) + (m-1)z(n,m-1,\xi)\phi(n,m-1,\xi)\cdot\mathbf{1}\{m>1\} \\ &+ \iota_{\xi'\to\xi}\phi(n,m,\xi') + \tilde{\beta}e\cdot\varrho(n)\cdot\mathbf{1}\{m=1\}, \end{aligned} \tag{35}$$

subject to the normalization $\sum_{\xi\in\{\xi_L,\xi_H\}}\sum_{m=1}^{\overline{m}} m\int_{\underline{n}}^{\infty}\phi(n,m,\xi)dn = 1$, where $\mathcal{S}_t(n,m,\xi)$ is the measure of firms with number of products m and innovation productivity type ξ that attain a resource level above n after matching with the financier at time t, and $\tilde{\beta} := \beta\cdot\mathbf{1}\{\xi = L\} + (1-\beta)\cdot\mathbf{1}\{\xi = H\}$ is the probability of an entrant being of type ξ.

The first term of equation (35) captures inflows and outflows due to continuous movements in net worth n. The second term accounts for inflows and outflows due to jumps in net worth from equity financing, while the third one includes outflows due to incumbent innovation, innovation productivity type transitions and displacement effects. The fourth term captures inflows from innovations of firms of size $m-1$, the fifth one represents inflows from type transitions, and the sixth term accounts for inflows from entrants displacing incumbents.

3.2.6 Growth

In a BGP equilibrium, all aggregate variables grow at the same constant rate. Then, by equation (33), $g \equiv \dot{Y}_t/Y_t = \dot{Q}_t/Q_t = \dot{K}_t/K_t$, and by the resource constraint, $g = \dot{C}_t/C_t = r - \rho$. In turn, the evolution of Q_t can be characterized as in Appendix B.4, where we show that:

$$g = \ln(\lambda)\left(\int_0^1 z_{jt}\,dj + e\right) \equiv \ln(\lambda)\tau. \tag{36}$$

The aggregate growth rate of the economy is therefore proportional to the aggregate rate of creative destruction, τ, which is driven by: (i) the expansion of existing firms into new markets, and (ii) the entry of new producers into the economy.

3.2.7 Welfare

Along a balanced growth path, consumption C_t grows at the same rate as Y_t. Therefore, the welfare of the representative household can be summarized by C_0 and g:

$$W = \frac{\ln(C_0)}{\rho} + \frac{g}{\rho^2}, \tag{37}$$

where initial consumption, C_0, is given by the aggregate resource constraint evaluated at $t = 0$:

$$C_0 = Y_0 \left(1 - \int_0^1 \xi z_{j0}^{\psi} m_{j0}^{\sigma-1} \, dj - \tilde{\xi} e_0^{\psi}\right) - \Sigma_0 - (g + \delta) K_0. \tag{38}$$

To compute initial output, Y_0, in turn, we normalize $Q_0 = 1$, which allows us to pin down w_0 from equation (32). Given w_0, we then solve for initial output through labor market clearing $\int_0^1 (l_{ijt} + l_{-ijt}) dj = 1$.[16]

Equation (37) shows that financial frictions affect welfare through both a level and a dynamic effect. The first term captures the static effect: financial frictions distort the static allocation of production, aggregate output, and hence consumption. The second term captures the dynamic welfare losses due to lower R&D investments and hence lower productivity growth, g. Because financial frictions also reallocate resources across consumption, investment, and innovation, the net effect on the level of initial consumption, and so welfare, is theoretically ambiguous.

We compute the consumption-equivalent welfare change between two BGP equilibria (labeled A and B) as the percentage change in lifetime consumption that the representative consumer of economy "A" would require to remain indifferent between the two BGPs:

$$\frac{1}{\rho}\left(\ln(C_0^B) + \frac{g^B}{\rho}\right) = \frac{1}{\rho}\left(\ln\left(C_0^A(1+\iota)\right) + \frac{g^A}{\rho}\right).$$

Solving for ι:

$$\iota = \frac{1}{C_0^A} \exp\left[\ln(C_0^B) + \frac{g^B - g^A}{\rho}\right] - 1.$$

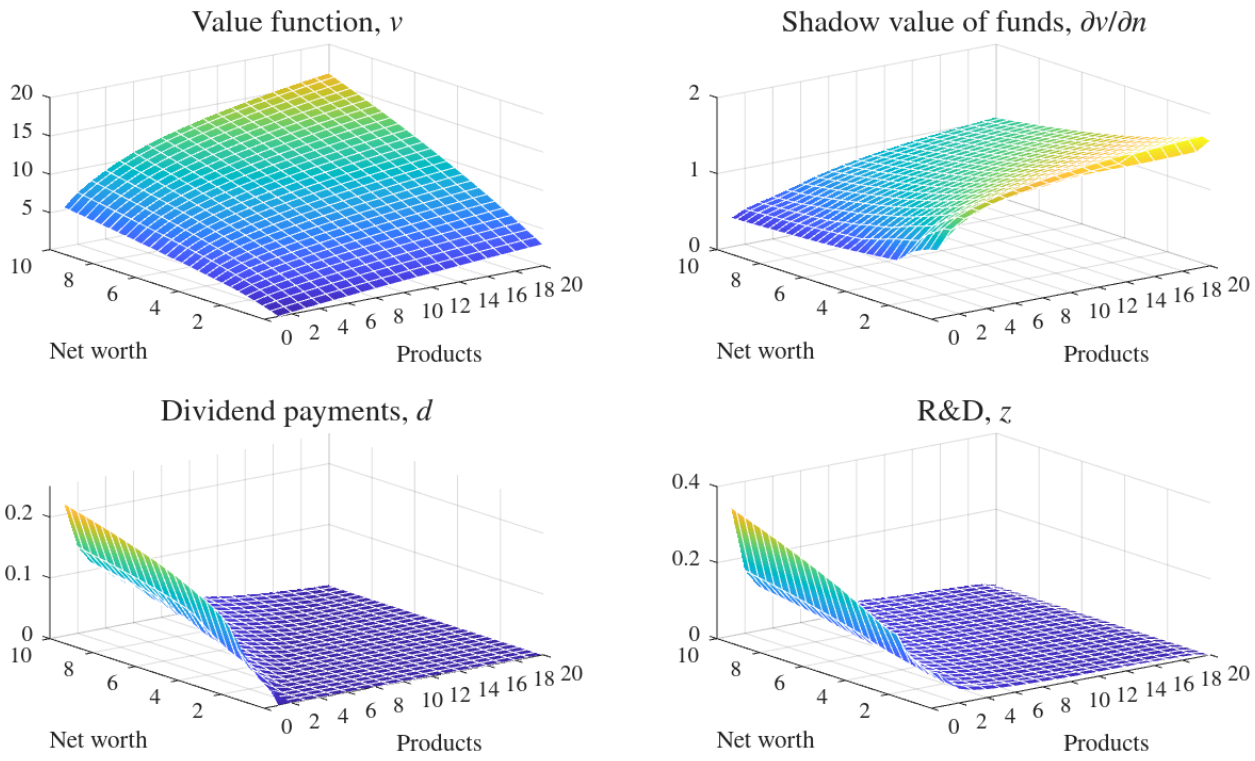
3.3 Model Properties

Figure 4 plots the value function, its derivative with respect to net worth, and the policy functions for dividend payments and R&D intensity, as functions of net worth and the number of product lines. Firm value is increasing and concave in both arguments, and

[16] Further details are provided in Appendix B.6.

flatter in m for low values of net worth. The shadow value of funds is largest for firms with low net worth per product line, which face a binding collateral constraint that limits the amount of capital they can use in production and reduces their stream of profits. For a given m, as net worth increases, the marginal value of internal funds falls as the borrowing constraint relaxes, because firms have more assets to pledge. Dividend payments are zero throughout the region in which the collateral constraint binds: constrained firms retain all of their earnings in order to rebuild net worth, and begin to pay out only once net worth per product line is high enough that the constraint is slack. As shown in the bottom-right panel, financial frictions depress R&D investments for financially constrained firms.

Figure 4: Value function, shadow value of funds, payouts and innovation policies.

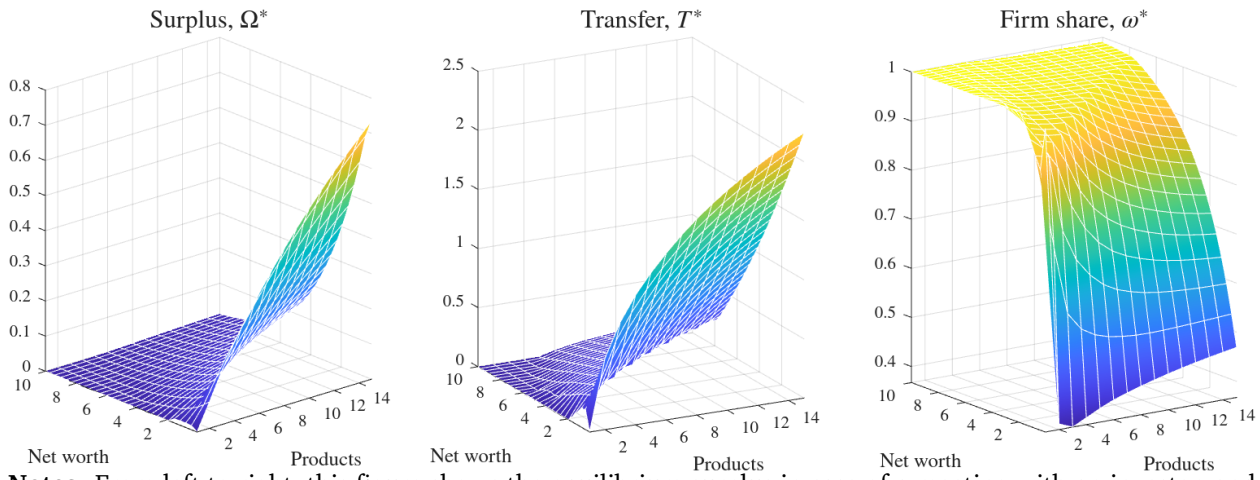


Notes: The top panels plot the value function and the shadow value of net worth; the bottom panels plot the policy functions for dividend payments and R&D intensity. All are shown as functions of net worth n and the number of product lines m, under the parameterization in Table 3. Surfaces average the two innovation types, weighted by their shares in the stationary distribution.

Figure 5 displays equilibrium objects in the equity financing market. The flat regions in the three panels are states in which no issuance takes place: the surplus is nil, no transfer is made, and the firm retains full ownership. For financially constrained firms, by contrast, the high shadow value of funds makes the surplus from an equity injection large, as shown in the left panel. For a given level of net worth, the surplus is zero until the firm operates enough product lines for the collateral constraint to bind, and rises steeply thereafter as the

firm moves further from unconstrained production. The middle and right panels describe the terms of the injection. Firms deep in the constrained region receive the largest transfers and, correspondingly, give up the largest ownership stakes to outsiders.

Figure 5: Outside equity: surplus, transfers and shares sold.



Notes: From left to right, this figure shows the equilibrium surplus in case of a meeting with an investor, and the implied transfer and ownership share kept by the firm under the parameterization in Table 3.

4 Calibration

This section details the calibration of the model and assesses its performance in reproducing key empirical regularities. We calibrate the model to the sample of French innovative firms. We discipline the model's parameters using a combination of externally set values and internally calibrated moments. Last, we assess the model's fit by examining how well it reproduces both the targeted moments and a broader set of empirical facts.

4.1 Externally Calibrated Parameters

We set six parameters externally: $(\rho, \alpha, \delta, \psi, \sigma, \eta)$. We fix the discount rate to $\rho = 0.02$, which together with a 1.3% target for the growth rate, implies a 3.3% annualized interest rate. We set the capital elasticity of output α to 0.33 and the depreciation rate of capital δ to 0.06. We set the curvature parameter of the R&D cost function to $\psi = 2$, following Akcigit and Kerr (2018) and prior empirical literature estimating the elasticity of innovation to R&D. Together with Akcigit and Kerr (2018)'s estimate of 0.395 for the degree of decreasing returns to firm size for external innovation, this implies $\sigma = 1.21$. Finally, we set the firm's weight in the bargaining with equity investors, η, to 0.5.

4.2 Internally Calibrated Parameters

The remaining twelve parameters,

$$(\lambda, \xi_L, \xi_L/\xi_H, \tilde{\xi}, \iota_{L\to H}, \iota_{H\to L}, \upsilon, \theta_n, \theta_y, \kappa, \chi, \zeta),$$

are calibrated internally to minimize the distance between model-generated moments and their empirical counterparts. Table 3 reports the results of the calibration in terms of model fit and provides the corresponding parameter values for the French economy.

Innovation and growth The first target is the growth rate of GDP per capita, which helps us discipline the innovation step size λ. We target France's average GDP per capita growth rate of 1.3 percent over the period 1980–2019 (World Bank). The R&D cost shifter of low-type incumbents, ξ_L, is pinned down by the R&D share of GDP (World Bank, average 1996-2019). The ratio ξ_L/ξ_H is identified by the patenting rate, defined as the fraction of firm-year observations in which the firm applies for at least one patent. The transition rates between types, $\iota_{L\to H}$ and $\iota_{H\to L}$, are disciplined by the empirical state-transition dynamics of firm patenting activity. Specifically, we match the conditional probabilities that a firm applies for at least one patent in year t, given that it did not apply for any patents in period $t-1$, $P(patent_t = 1|patent_{t-1} = 0)$, and given that it did apply for at least one patent in period $t-1$, $P(patent_t = 1|patent_{t-1} = 1)$.[17] Finally, to discipline the scale parameter of entrants' R&D cost function, $\tilde{\xi}$, we use the firm exit rate.

Finance The parameter θ_n of the collateral constraint is disciplined by aggregate leverage, defined as the ratio of net financial debt to financial capital. Regarding θ_y, following Jeenas, Pijoan-Mas, Moral-Benito, Garcia-Santana and di Giovanni (2022), we proceed as follows. We start by rewriting the borrowing constraint in (11) in first differences and relative to capital k_t, such that the change in leverage for financially constrained firms is given by:

$$\Delta\left(\frac{b_t}{k_t}\right) = \vartheta_y \Delta\left(\frac{\sum_{j\in\mathcal{M}_t} p_{jt}y_{jt}}{k_t}\right) \tag{39}$$

We use this equation to recover ϑ_y by indirect inference. Specifically, we require the OLS estimation of equation (39) to deliver the same regression coefficients in the model and in the data. In the data, we construct the right-hand side variable of (39) as the change in

[17] The estimated $\iota_{L\to H}$ and $\iota_{H\to L}$ imply that the stationary share of low-type firms from the Markov chain described by the transitions across types (β) is 83.5%.

total value added over the firm's capital stock, and estimate this equation with industry-year and firm fixed effects on the sample of firms with leverage ratios above the median.

We choose the parameter v of the distribution of entrants' initial equity to match the average net worth of entrants relative to the average net worth of all firms. The penalty on dividend adjustment, κ, is chosen to match the profit-weighted average dividend payout ratio in the data.[18] We also target the frequency of equity injections, which identifies the parameter χ, governing the rate at which firms meet investors. Conditional on an equity injection, the parameter ζ governs the size of the issuance. Therefore, we pin down this parameter to match the average size of an equity injection, relative to firm's equity, conditional on a positive equity injection.

Table 3: Parameters and model fit.

Parameter	*Description*	*Value*	*Target/Source*	*Data*	*Model*
	Panel A: *Externally Calibrated Parameters*				
ρ	Discount rate	0.02	Standard value		
α	Capital elasticity of output	0.33	Capital share of GDP		
δ	Depreciation rate of capital	0.06	Standard value		
ψ	R&D cost curvature	2	Akcigit and Kerr (2018)		
σ	Elasticity of R&D cost w.r.t. firm size	1.21	Akcigit and Kerr (2018)		
η	Firm's bargaining weight	0.5	Assumption		
	Panel B: *Internally Calibrated Parameters*				
λ	Innovation step size	1.165	Growth rate	1.3%	1.3%
ξ_L	R&D scale, incumbents L-type	294.64	R&D share of GDP	2.2%	2.2%
ξ_L/ξ_H	R&D scale, incumbents L/H	491.7	Patenting rate	12.7%	12.8%
$\tilde{\xi}$	R&D scale, entrants	41.5	Exit rate	3.8%	4.5%
$\iota_{L\to H}$	$L \to H$ transition rate	0.021	$P(patent_t = 1 \mid patent_{t-1} = 0)$	6.2%	6.0%
$\iota_{H\to L}$	$H \to L$ transition rate	0.106	$P(patent_t = 1 \mid patent_{t-1} = 1)$	55.5%	59.0%
v	Rate exponential initial equity entrants	0.8175	Avg. net worth 0–2 yrs rel. to all firms	23.1%	19.5%
κ	Dividend adjustment penalty	10.443	Avg. dividend payout ratio	96.0%	99.0%
θ_n	Collateral multiplier, net worth	1.4545	Avg. leverage	28.5%	30.6%
θ_y	Collateral multiplier, revenues	0.095	Leverage reg. coefficient	0.063	0.063
χ	Meeting rate, investors	0.531	Frequency of equity injections	8.4%	7.8%
ζ	Outside equity cost scale	0.187	Average size equity injection	14.4%	12.8%

Notes: The model period is one year. Panel A reports externally calibrated parameters. Panel B reports internally calibrated parameters. The growth rate and the R&D share of GDP are from the World Bank. Remaining targets in Panel B are estimated from ORBIS. All ratios are weighted by the denominator variable.

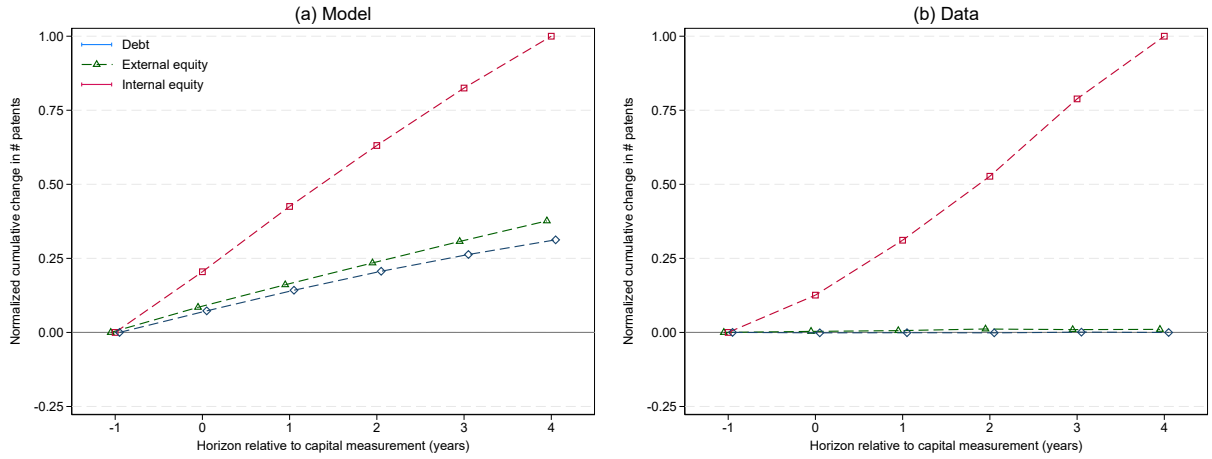
[18] We do not observe dividend payments directly in our firm-level data, and therefore we infer them from changes in internal equity, which reflect retained after-tax profits net of distributions. Specifically, we construct dividend payments as after-tax profits minus the change in Other Shareholder Funds.

4.3 Model Validation

To validate our calibration strategy, we compare the effects of financial-capital injections on innovation in the model and in the data. Figure 6 shows the results of estimating equation (3) on model-generated data alongside the corresponding empirical estimates for French patenting firms.

In both the model and the data, external equity injections are associated with larger increases in patenting, whereas debt and internal equity display a more muted relationship with innovation outcomes.[19] In our model, firms are indifferent to the source of net worth: their R&D activity is solely driven by their net worth and their number of products. The stronger correlation between external equity and patenting reflects selection: financiers invest in firms with higher returns to net worth. As shown in Sections 3.2.2 and 3.3, these firms are typically financially constrained and face both higher marginal costs and lower marginal benefits of doing R&D. For them, an increase in net worth generates a larger increase in innovation than for unconstrained firms, which have larger self-financing capacity.

Figure 6: Untargeted moments.



Notes: Estimates from running equation (3) for model-generated data under the parameterization in Table 3 (in levels), and in the data for the sample of French patenting firms. The dependent variable in the model-generated data is the cumulative number of products developed. The vertical axis is normalized with respect to the largest positive value in the each plot.

In the next section, we quantify the aggregate effects of financial frictions on productivity

[19] In the model, innovation output is measured by the number of developed products, which we map to patents in the data. The R&D process requires successful patenting to introduce a new product, and patent protection grants temporary leadership in the product market.

growth, and study their implications for policy.

5 Financial Frictions and Growth

We now tackle the main research questions of our paper: how large are the dynamic output losses due to financial frictions? To what extent can equity markets mitigate the productivity growth losses due to credit frictions? What are the implications for the design of growth-stimulating policies?

5.1 The Role of Debt and Equity Frictions

We assess the growth implications of relaxing or tightening firms' access to external finance by performing a set of counterfactual exercises. The results are reported in Table 4.

We begin by examining the role of debt market frictions. Column (2) of Table 4 considers an economy in which firms face no collateral constraint on borrowing, implemented by letting the collateral parameters, θ_n and θ_y, tend to infinity. Relative to the baseline economy in Column (1), the growth rate is 1.25 percentage points higher: removing borrowing constraints doubles the rate of aggregate productivity growth. The gain accrues overwhelmingly to incumbents: their contribution to aggregate growth rises from 1.05 to 2.18 percentage points, accounting for roughly 90% of the total increase, while the entrants' contribution rises from 0.21 to 0.33. The fall in the entry share therefore reflects faster incumbent innovation rather than weaker entry. Average leverage rises from 31% to 97%, while average net worth collapses. Once borrowing is unconstrained, retained earnings lose their shadow value as a means of relaxing future constraints, so firms distribute them and finance production almost entirely with debt. For the same reason, no firm relies on external equity: issuance frequency and the average equity transfer both fall to zero. Finally, the share of product lines held by high-innovation-capacity (H-type) firms rises from 43.4% to 57.2%, indicating that easier credit reallocates production toward the most capable innovators.

Column (3) performs the opposite exercise, making debt frictions as severe as possible ($\theta_n \to 1$, $\theta_y \to 0$), so that firms can no longer borrow at all and aggregate leverage falls to zero. The collateral multiplier rises sharply relative to the baseline, reflecting a marked tightening of borrowing constraints across the firm distribution. Firms respond by reducing production and innovating less, as reflected in the lower average number of products, and by accumulating more net worth. External equity issuance frequency and the average transfer fall, so that the increase in net worth is due to larger retained earnings, consistent

with a strengthened precautionary motive. Growth falls by 24 basis points. Both the incumbents and the entry margin contract in levels, so that the rise in the entry share of growth reflects the sharper decline of incumbent innovation.

How much do equity markets mitigate the aggregate productivity growth losses caused by debt frictions? To answer this question, we compare the baseline economy to an economy with no external equity markets. Column (4) shows that when equity markets are shut down ($\chi = 0$), growth falls by 25 basis points. Shutting down equity markets is therefore slightly more costly than shutting down debt markets. Absent equity markets, the growth losses due to debt frictions would therefore be about 20% larger. When external equity cannot be raised, firms end up both poorer and smaller: average net worth and the average number of products both decline. Lower net worth in turn tightens the collateral constraint, as shown by the increase in the collateral multiplier. The reallocation away from H-type firms is larger without equity markets than without debt markets: the share of product lines they hold falls to 37.9% in Column (4), against 39.7% in Column (3).

Table 4: Macroeconomic effects of financial frictions.

		Debt frictions		Equity markets		Equity: decomposition	
	(1) Baseline	(2) $\theta_n \to \infty$, $\theta_y \to \infty$	(3) $\theta_n \to 1$, $\theta_y \to 0$	(4) $\chi \to 0$	(5) $\zeta \to 0$, $\chi \to \infty$	(6) $\zeta \to 0$	(7) $\chi \to \infty$
Growth rate (%)	1.26	2.51	1.02	1.01	1.43	1.38	1.42
change vs. baseline (b.p.)	—	+125.5	−23.5	−25.3	+16.8	+12.5	+16.4
Incumbents share (%)	83.64	86.70	82.55	77.52	85.37	85.22	85.37
Entry share (%)	16.36	13.30	17.45	22.48	14.63	14.78	14.63
Avg. leverage	0.31	0.97	0.00	0.31	0.31	0.31	0.31
Avg. collateral multiplier	17.26	0.00	25.68	17.74	29.04	27.32	29.73
Equity issuance freq.	0.08	0.00	0.05	0.00	0.00	0.03	0.01
Avg. rel. equity transfer	0.13	0.00	0.10	0.00	0.01	0.40	0.00
Avg. net worth	6.26	0.36	6.91	5.30	6.65	6.69	6.65
Avg. number of products	3.39	4.09	3.23	2.76	3.70	3.66	3.70
Share of lines held by H-type (%)	43.44	57.20	39.71	37.89	46.25	45.65	46.20

Notes: BGP equilibrium values in the baseline calibration and in counterfactual economies with extreme values of the parameters governing debt and equity financing frictions. Columns (2)–(3) vary debt frictions; Columns (4)–(5) shut down and perfect equity markets respectively; Columns (6)–(7) decompose the effect in Column (5) into its issuance-cost and search components.

What if equity markets were frictionless? In Column (5), we consider an economy with no search frictions in equity markets ($\chi \to \infty$) and no equity issuance costs ($\zeta = 0$). Growth rises by 17 basis points relative to the baseline, almost entirely through incumbent innovation. Perfectly functioning equity markets would therefore close 13% of the growth gap created by borrowing constraints. Columns (6) and (7) show that removing search

frictions has a larger quantitative effect on growth than removing equity issuance costs alone: eliminating issuance costs raises growth by 12.5 basis points, while eliminating search frictions alone raises it by 16 basis points, accounting for almost all of the gain in Column (5). The reason is visible in the equity moments. When firms can access outside investors instantaneously, they recapitalize once and thereafter have no further use for the equity market: the frequency of equity issuance falls from 0.08 in the baseline to 0.01 in Column (7). With almost no issuance taking place in the stationary equilibrium, the issuance cost applies to a vanishing base. Issuance costs matter, in other words, only insofar as search frictions force firms to raise equity at infrequent and discrete opportunities.

Taken together, these counterfactuals reveal that equity finance plays a meaningful role in mitigating the effects of debt frictions. Debt frictions reduce static profits, slow down internal equity accumulation, and strengthen future borrowing constraints. These effects operate at all stages of the firm's life cycle and are particularly powerful for small firms, whose financial tightness limits their ability to scale up production. Equity finance is able to offset a non-trivial share of these losses primarily by relaxing the financing constraints of incumbents.

5.2 Policy

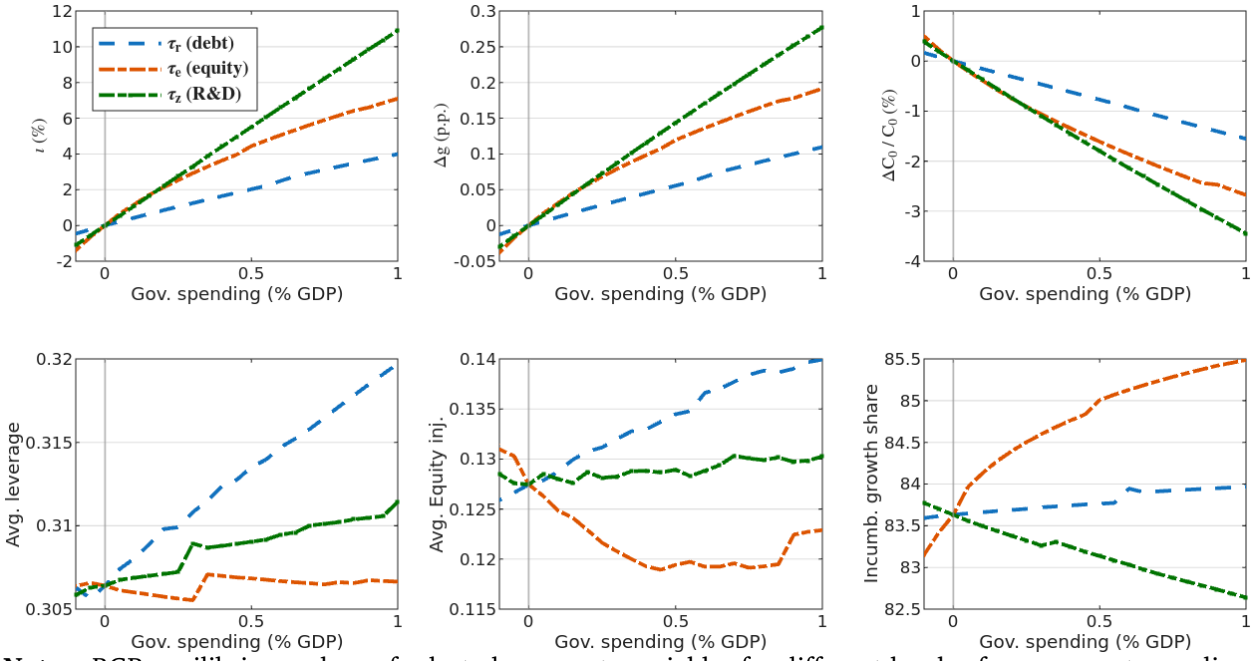
The previous results show that imperfect access to external finance discourages firm-level innovation and weakens aggregate growth. We now ask whether policies that reduce the cost of finance can stimulate R&D, and what their aggregate consequences are.

We evaluate two finance-related instruments—a subsidy to interest payments on debt and a subsidy to the cost of capital injections from outside investors—and compare them with a standard R&D subsidy. In each experiment, the government pays a fraction of the relevant cost and finances the outlay through a lump-sum tax on the representative household.[20]

Figure 7 reports the effects of the three subsidies on selected BGP variables, for different levels of government spending. We first focus on welfare and growth, shown in the top row. At low levels of spending, the equity subsidy delivers the largest welfare gains because it directs resources toward firms with a high shadow value of funds. At a level of government spending equal to 0.1% of GDP, consumption-equivalent welfare rises by 1.2% under the equity subsidy, versus 1.1% under the R&D subsidy. As the subsidy expands and cheaper

[20] Under the debt subsidy, a firm with positive debt pays $b_t r_t(1-\tau_r)$; under the equity subsidy, investors pay $T_t(1-\tau_e)+\frac{\zeta}{2}T_t^2$ for an injection T_t, so that only the linear transfer is subsidized; under the R&D subsidy, firms pay $(1-\tau_z)c_t^z(z,m)$.

Figure 7: Effects of government subsidies on BGP equilibrium variables.



Notes: BGP equilibrium values of selected aggregate variables for different levels of government spending. Each line represents a different economy. The vertical gray line marks the benchmark calibrated economy. Consumption-equivalent welfare is computed as described in Section 3.2.7.

equity is increasingly allocated to firms with lower returns to external funding, the R&D subsidy dominates. At a level of government spending equal to 0.5% of GDP, welfare rises by approximately 5.5% under the R&D subsidy, versus 4.4% under the equity subsidy. The debt subsidy yields the smallest welfare gains throughout the range we study. The same pattern holds for growth, though less sharply: the equity and R&D subsidies are essentially tied at the lowest spending levels and the R&D subsidy dominates thereafter (1.40% versus 1.38% at 0.5% of GDP). The welfare ranking reflects two opposing forces. On the one hand, subsidies raise growth. On the other hand, they induce firms to devote more real resources to R&D, which crowds out initial consumption. At low levels of spending the equity subsidy generates slightly more growth than the R&D subsidy, while the two policies' costs in terms of initial consumption are nearly identical. As spending rises, the growth return to the equity subsidy flattens while that of the R&D subsidy remains close to linear, and the R&D subsidy overtakes it. Beyond that point the R&D subsidy also carries the larger initial-consumption cost, because it induces a larger increase in real R&D spending, but not by enough to offset its growth advantage.

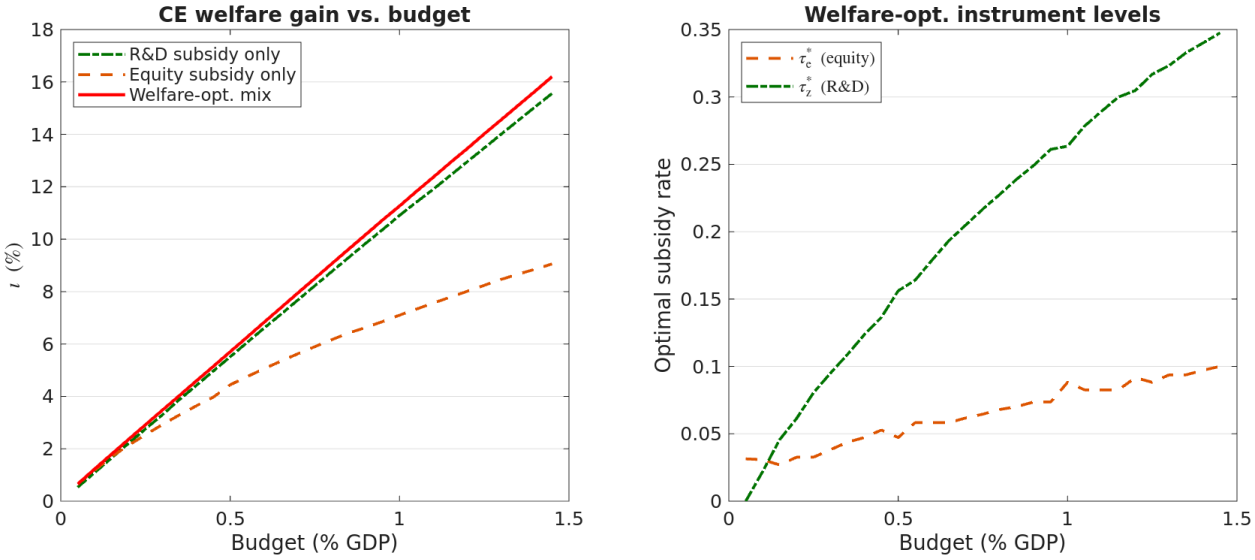
The bottom row shows how each instrument operates. The debt subsidy raises average leverage, while the other two instruments leave it nearly unchanged. The middle panel reports the average equity injection relative to the net worth of the firms receiving it. The

equity subsidy lowers it modestly, even though it roughly doubles the size of injections: the net worth of the firms receiving equity rises by more than the transfers they receive. Finally, incumbents' share of growth rises under the equity subsidy and falls under the R&D subsidy, which shifts innovation toward entrants.

Because the welfare ranking of the individual instruments varies with the level of spending, no single instrument is optimal over the entire range. We therefore consider the welfare-maximizing combination of the two strongest instruments: the external-equity and R&D subsidies.

Figure 8 reports the welfare-maximizing combination of external-equity and R&D subsidies. By construction, the policy mix weakly dominates either instrument in isolation, as shown in the left panel. The right panel reports the subsidy rates that implement it. At the smallest budgets, the optimal policy is equity-only; at higher levels of spending, it places positive weight on both instruments and it is tilted toward R&D. The R&D subsidy rate rises steeply with the budget while the equity subsidy rate increases far more slowly, so that the ratio between the two settles at roughly three to one. At government spending equal to 1% of GDP, the optimal mix combines an R&D subsidy of 26.3% with an equity subsidy of 8.8%, raising the growth rate from 1.26% to 1.54% and consumption-equivalent welfare by 11.3% relative to the baseline.

Figure 8: Optimal government expenditure on different subsidies.



Notes: The left panel reports consumption-equivalent welfare gains relative to the baseline economy against government spending as a share of GDP, under three policies: an R&D subsidy only, an external-equity subsidy only, and the welfare-maximizing combination of the two. The right panel reports the subsidy rates that implement the optimal policy. Consumption-equivalent welfare is computed as in Section 3.2.7.

The gains from combining the two instruments are nonetheless modest. Equity-only

is the better single instrument only for budgets below roughly 0.2% of GDP; beyond that point the R&D subsidy dominates, and at 1% of GDP it delivers on its own a welfare gain of 10.9%, within half a percentage point of the optimal mix. The equity subsidy therefore complements rather than substitutes for the R&D subsidy in the optimal policy. Lowering the cost of external equity channels resources to firms with a high shadow value of funds, which raises the return to subsidizing R&D precisely for those firms whose innovation is limited by internal finance rather than by the cost of innovating.

6 Conclusions

In this paper, we study how access to external finance shapes innovation and the scale-up of innovative firms. Using data on European patenting firms, we show that equity and debt reach systematically different firms: external-equity inflows are directed toward smaller, younger, collateral-poor, and faster-growing firms, whereas debt reaches larger, more collateral-rich firms. Moreover, capital inflows are associated with subsequent increases in investment and patenting, and this association is strongest when the source of capital is external equity. We interpret these findings as evidence of imperfect access to external finance, with equity reaching the firms whose innovation is most sensitive to financial capital. Motivated by these empirical facts, we build a new endogenous growth model of firm innovation and finance to study the aggregate consequences of financial accessibility for economic growth and creative destruction.

In our model, constraints to firms' access to external finance affect growth through two channels. First, financially constrained incumbents face a high shadow value of funds and low returns from expanding into new product lines, leading them to keep resources inside the firm and reducing their R&D investments. It is precisely these firms, whose shadow value of funds is highest, that outside investors fund via external equity the most. Second, financial frictions depress firm value upon entry, reducing the rate of new firm creation. Our counterfactual exercises highlight the quantitative importance of these mechanisms, showing that financial frictions have first-order consequences for innovation and long-run growth. In the perfect credit benchmark, growth is 1.25 percentage points higher than in the baseline, driven predominantly by higher incumbent innovation. Equity finance mitigates part of this loss: absent search frictions and issuance costs, it would close 13% of this gap.

Finally, we use the model to evaluate policies aimed at stimulating innovation and growth, comparing subsidies to debt, external equity, and R&D. At low levels of spending, the equity subsidy delivers the largest welfare gains, as it channels resources to firms with a

high shadow value of funds. As equity is increasingly allocated to firms with lower returns to funding, the R&D subsidy becomes the most effective. The welfare-maximizing policy subsidizes only external equity at the smallest government budgets and tilts toward R&D as the budget grows. At government spending equal to 1% of GDP, the optimal mix raises consumption-equivalent welfare by 11.3% relative to the baseline. Our findings point to complementary roles for equity-market development and R&D support in fostering the scale-up of innovative firms and stimulating innovation-led growth.

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Financing the Scale-Up: Equity, Innovation and Economic Growth

by Andrea Guccione and Juan Segura

Appendix Materials

A Data Appendix

A.1 Sample Selection and Data Cleaning

This section discusses the sample selection and cleaning of Orbis Historical data.

Sample selection For firms that report both consolidated and unconsolidated accounts in a given year, we keep unconsolidated accounts only. Moreover, we only keep firms with the following legal statuses: Partnerships, Private limited companies, Public limited companies, Sole traders/proprietorships. Finally, we exclude firms that operate in: finance; insurance; real estate; public administration and defense; education; healthcare and social work; membership organizations; activities of households as employers, undifferentiated goods and services; activities of extraterritorial organizations and bodies. We also exclude firms with missing NACE information at the 2-digit level.

Data cleaning We implement the following cleaning procedure on the balance sheet and income statement data, following Kalemli-Özcan *et al.* (2024):

- Drop company-years with missing information on total assets and operating revenue and sales and employment (simultaneously).
- Drop the entire company (all years) if total assets is negative in any year.
- Drop the entire company if employment (in persons) is negative in any year and companies with employment larger than that of Walmart (2 million) in any year.
- Drop the entire company if sales are negative in any year.
- Drop the entire company when reporting in any year a value of employment per million of total assets larger than the 99.9 percentile of the distribution.
- Drop the entire company when reporting in any year a value of employment per million of sales larger than the 99.9 percentile of the distribution.
- Drop the entire company when reporting in any year a value of sales to total assets larger than the 99.9 percentile of the distribution.
- Drop the entire company if Tangible Fixed Assets is negative in any year.

A.2 Variable Definitions

- Financial debt: sum of short-term and long-term financial debt (`tdb` = `loan` + `ltdb`).
- Net financial debt: financial debt minus cash (`b` = `tdb` - `cash`).
- External equity: `capi`.
- Internal equity: `osfd`.
- Equity (net worth): `capi` + `osfd`.
- Financial capital: `FC` = `tdb` + `capi` + `osfd`.
- Net financial capital: `b` + `capi` + `osfd`.
- Injection from source $X \in \{\texttt{capi}, \texttt{osfd}, \texttt{tdb}\}$: an increase in the component of more than 1%, $\mathbf{1}\{\Delta X_{it} > 0.01\,|X_{it-1}|\}$.
- Size of an injection from source X, relative to financial capital: $\Delta X_{it}/\texttt{FC}_{it-1}$, computed conditional on an injection from source X.
- Size of an equity injection, relative to total equity: $\Delta\texttt{capi}_{it}/(\texttt{capi}_{it} + \texttt{osfd}_{it})$, computed conditional on an equity injection.
- Dividends: after-tax profits minus change in internal equity ($\texttt{plat} - \Delta\texttt{osfd}_{it}$).

A.3 Additional Empirical Results

Summary statistics Table A.1 reports summary statistics for the sample of patenting firms.

Capital structure We provide descriptive statistics on firms' financial structure, distinguishing between innovative and non-innovative firms. Table A.2 provides a decomposition of a firm's financial capital into its three components. Panel A reports unconditional means, while Panel B reports residualized means from regressions controlling for age, size and industry-year-country fixed effects.

Compared to non-innovative firms, on average, innovative firms rely less on debt finance (34.3% vs. 44.6%), similarly on external equity (16.6% vs. 16.5%), and more on internally generated resources (49.1% vs. 38.9%).[21] Among the two sources of external finance, innovative firms raise a larger share of external funds from equity (32.6% vs. 27.0%). Overall, equity (internal plus external) accounts for

[21] This pattern is consistent with work showing that R&D-intensive firms maintain high cash holdings and precautionary liquidity buffers, reflecting the greater financing frictions associated with intangible and risky investment activities (Hall and Lerner, 2010; He and Wintoki, 2016).

Table A.1: Summary statistics.

	Mean	p10	p25	p50	p75	p90
Firm characteristics						
Age	25.0	4	10	20	33	49
Employees	213.4	2	8	25	81	240
Sales (€ k)	67,200	250	1,066	3,845	15,400	61,500
Value added (€ k)	20,700	155	465	1,554	5,856	19,200
Fin. capital (€ k)	16,300	101	402	1,694	7,347	29,500
Tang. assets (€ k)	4,689	14	67	356	1,830	7,670
Intang. assets (€ k)	860	1	6	37	214	1,114
Tangible share	0.865	0.622	0.823	0.953	0.992	0.999
Growth rates						
Employment	0.022	−0.162	−0.033	0.000	0.075	0.222
Sales	0.044	−0.281	−0.079	0.040	0.167	0.373
Innovation						
Patenting (rate)	0.125					
Legal status						
Public	0.171					

Notes: Sample of patenting firms in France, Germany, Italy and Spain, 1995–2023. Sales, value added, financial capital, and tangible and intangible assets are in thousands of euros and reported for firm-years with strictly positive values. The tangible share is tangible fixed assets over total fixed assets, weighted by lagged total fixed assets. Growth rates are annual DHS rates. Patenting is the share of firm-years in which the firm files at least one patent application. Public is the share of public limited companies.

a larger share of innovative firms' financial capital (65.7% vs. 55.4%).[22] Differences in age, size, and industry composition among innovative and non-innovative firms partly account for these differences in capital structure. However, innovative firms' stronger reliance on internal rather than external finance and on equity rather than debt still hold conditionally on these firm characteristics (Panel B).

Equity injections We now document facts about equity financing. Following Kochen (2023), we identify equity injections as episodes in which the external equity funds of a firm increase by more than 1%. When a firm receives an equity injection, we measure its size as the change in external equity relative to the firm's total financial capital.

In Table A.3, we report descriptive statistics about the frequency and size of equity injections, distinguishing between innovative and non-innovative firms. Unconditionally, equity injections are more common among innovative firms (6.5% of firm-year observations vs. 3.7%), which however receive smaller injections in these events (9.5% vs. 14.0% of financial capital for the average injection). The size gap reflects differences in firm characteristics: after controlling for age, size and industry-year-country fixed effects, innovative firms still receive equity injections more frequently (4.8% vs.

[22] Innovative firms' preference for equity finance over debt finance has also been documented in Brown, Fazzari and Petersen (2009), Hall and Lerner (2010) and Itenberg (2015).

Table A.2: Financial capital structure: innovative vs. non-innovative firms.

	Innovative	Non-innovative	**Difference**
Panel A: Unconditional			
Debt	34.3	44.6	**-10.3**
External equity	16.6	16.5	**0.1**
Internal equity	49.1	38.9	**10.2**
Panel B: Residualized			
Debt	38.8	42.6	**-3.8**
External equity	17.1	16.3	**0.8**
Internal equity	44.1	41.1	**3.0**
Observations	1,164,501	24,743,050	

Notes: Innovative firms are firms that have ever applied for a granted patent. Panel A reports unconditional means. Panel B reports residualized means from regressions controlling for age, size and industry-year-country fixed effects. Averages are weighted by financial capital.

3.8%), and the injections they receive are, if anything, slightly larger than those of non-innovative firms (13.6% vs. 12.5% of financial capital).

Table A.3: Equity injections: innovative vs. non-innovative firms.

	Innovative	Non-innovative	**Difference**
Panel A: Unconditional			
Frequency	6.5	3.7	**2.8**
Size, relative to financial capital	9.5	14.0	**-4.5**
Panel B: Residualized			
Frequency	4.8	3.8	**1.0**
Size, relative to financial capital	13.6	12.5	**1.1**
Observations (frequency/size)	1,805,001/97,832	48,618,882/1,354,982	

Notes: Innovative firms are firms that have ever applied for a granted patent. Panel A reports unconditional means. Panel B reports residualized means from regressions controlling for age, size and industry-year-country fixed effects. Average equity injection size is weighted by financial capital.

Selection To isolate the extensive margin of selection into each source of financing, we re-estimate (1) replacing the continuous inflows with injection indicators:

$$x_{i,t-1} = \theta^E \, \text{ieq}_{it} + \theta^L \, \text{idebt}_{it} + \theta^I \, \text{iosfd}_{it} + \gamma_{c(i)\,s(i)\,t} + u_{it}, \tag{A.1}$$

where each indicator equals one when the corresponding financial capital component rises by more than 1%. The estimation results are in Table A.4.

Net worth is strongly negative throughout: relative to no-event firm-years, equity-injection firms have about 0.08 SD lower net worth, 0.12 SD lower than debt-injection firms, and 0.15 SD lower

Table A.4: Selection on predetermined characteristics: extensive margin.

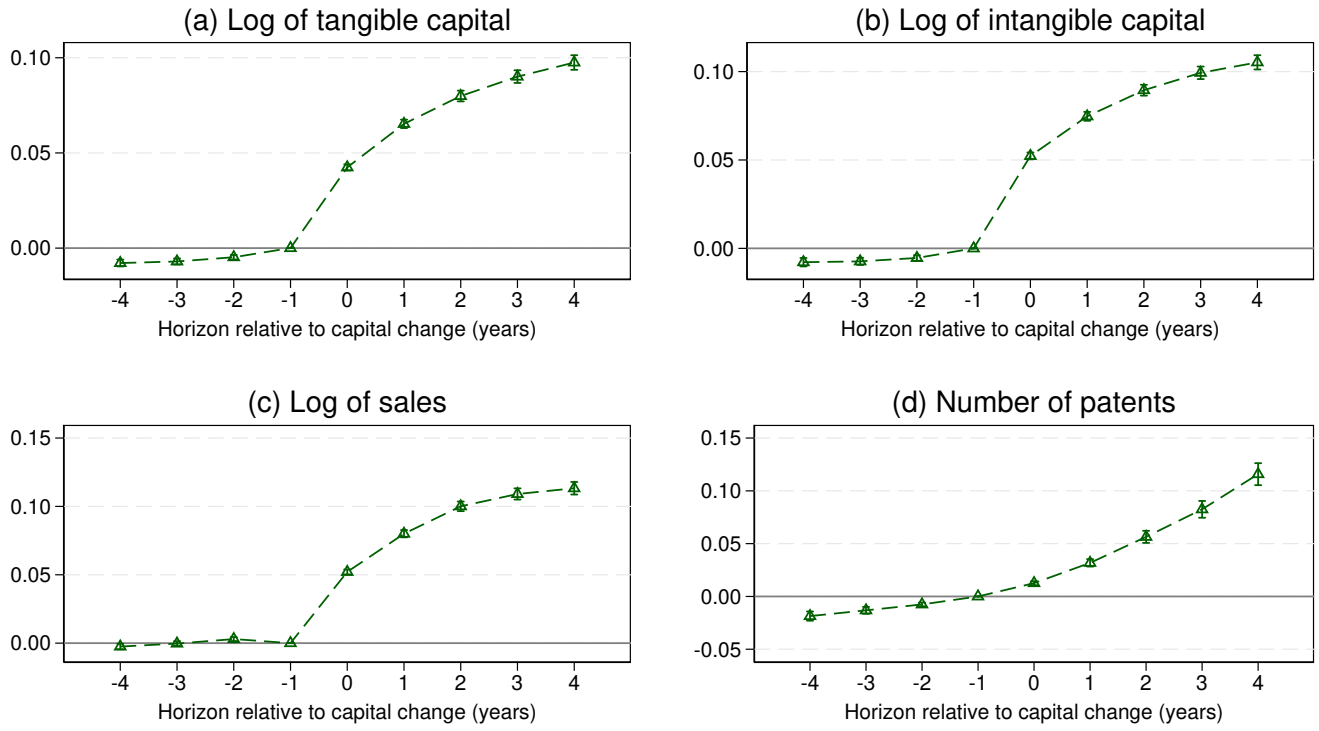
Characteristic ($t-1$)	External equity	Ext. eq. − Debt	Ext. eq. − Internal eq.
Log net worth	-0.077^{***}	-0.118^{***}	-0.146^{***}
	(0.005)	(0.005)	(0.005)
Log tangible capital	0.020^{***}	0.041^{***}	0.024^{***}
	(0.005)	(0.005)	(0.005)
Log total fixed capital	0.082^{***}	0.100^{***}	0.112^{***}
	(0.005)	(0.005)	(0.005)
Log sales	-0.042^{***}	-0.041^{***}	-0.144^{***}
	(0.006)	(0.006)	(0.006)
Log employment	0.028^{***}	0.025^{***}	0.132^{***}
	(0.006)	(0.006)	(0.006)
Age	-0.051^{***}	-0.021^{***}	0.071^{***}
	(0.007)	(0.007)	(0.008)
Prior sales growth	0.180^{***}	0.121^{***}	-0.009^{*}
	(0.005)	(0.005)	(0.005)
Prior employment growth	0.110^{***}	0.061^{***}	0.013^{**}
	(0.006)	(0.006)	(0.006)
Patenting	0.035^{***}	0.028^{***}	0.041^{***}
	(0.002)	(0.002)	(0.002)
Intangible share	0.043^{***}	0.037^{***}	0.053^{***}
	(0.001)	(0.001)	(0.002)

Notes: Each row regresses the predetermined ($t-1$) characteristic on the three injection indicators jointly. Col. 1: external-equity coefficient (vs. no-event firm-years); cols. 2–3: differences from the debt and internal-equity coefficients. The first eight characteristics are standardized to unit standard deviation; patenting and intangible share are in natural units. Country×industry×year FE; SE clustered by firm. $^{*}p<0.10$, $^{**}p<0.05$, $^{***}p<0.01$.

than firms with an internal-equity increase. Patenting and the intangible share are positive and equity-specific: equity recipients are about 3.5 pp more likely to be patenting and 4.3 pp more intangible-intensive than no-event firms, and more so than either debt or internal-equity firms. Prior growth is high for equity recipients (0.18 SD for sales, 0.11 SD for employment), and clearly higher than for debt-injection firms, but similar to that of internal-equity firms, as expected given that internal equity accumulates out of retained profits. The size and age tilt, by contrast, is largely an intensive-margin phenomenon: on the extensive margin, equity-injection firms are younger and have lower sales than no-event and debt-injection firms, but they hold more tangible and total fixed capital and employ more workers, and they are older and larger in employment than internal-equity firms. The large negative size loadings in Table 2 therefore reflect that the largest proportional injections accrue to small, young firms, rather than a difference in which firms receive an injection.

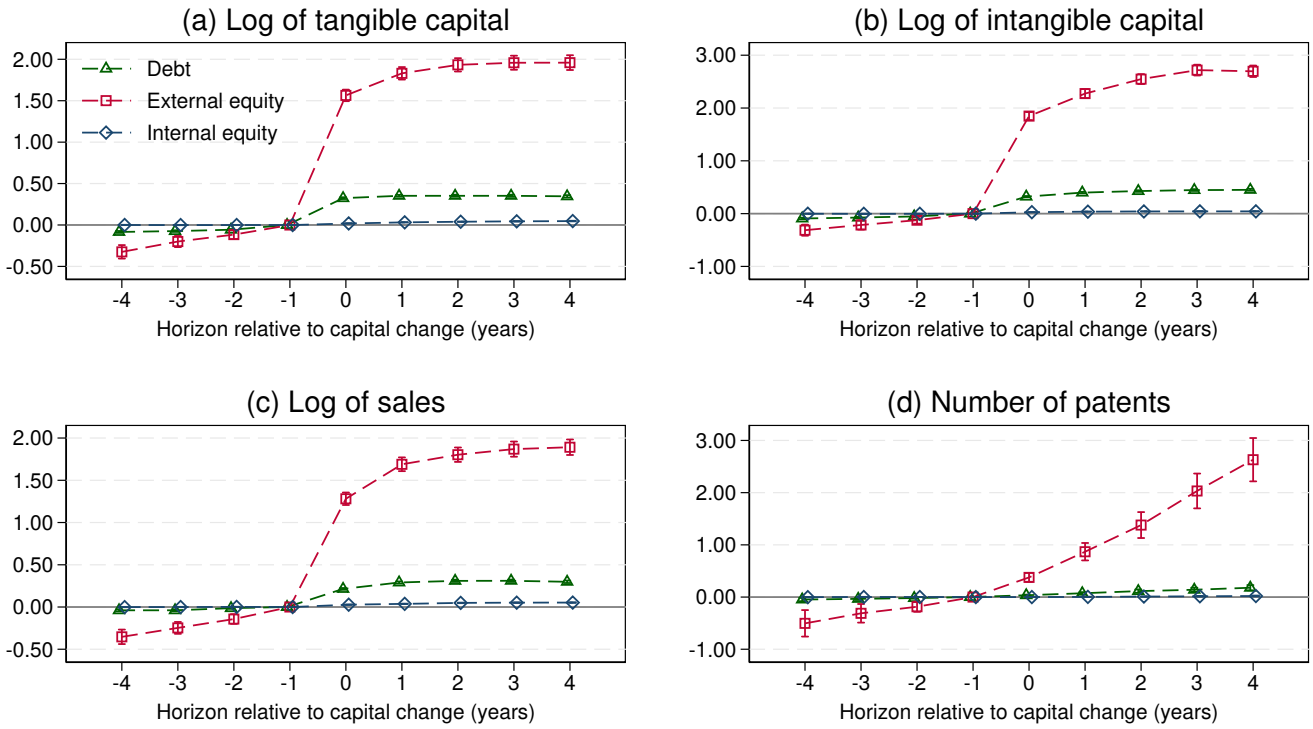
Local projections Figures A.1–A.4 report the local projection estimates for investment, sales and patenting, pooled and by financing source, and the sample splits by proxies for financial constraints for internal equity and debt, referred to in Section 2.

Figure A.1: Financial capital, investment, sales and patenting.



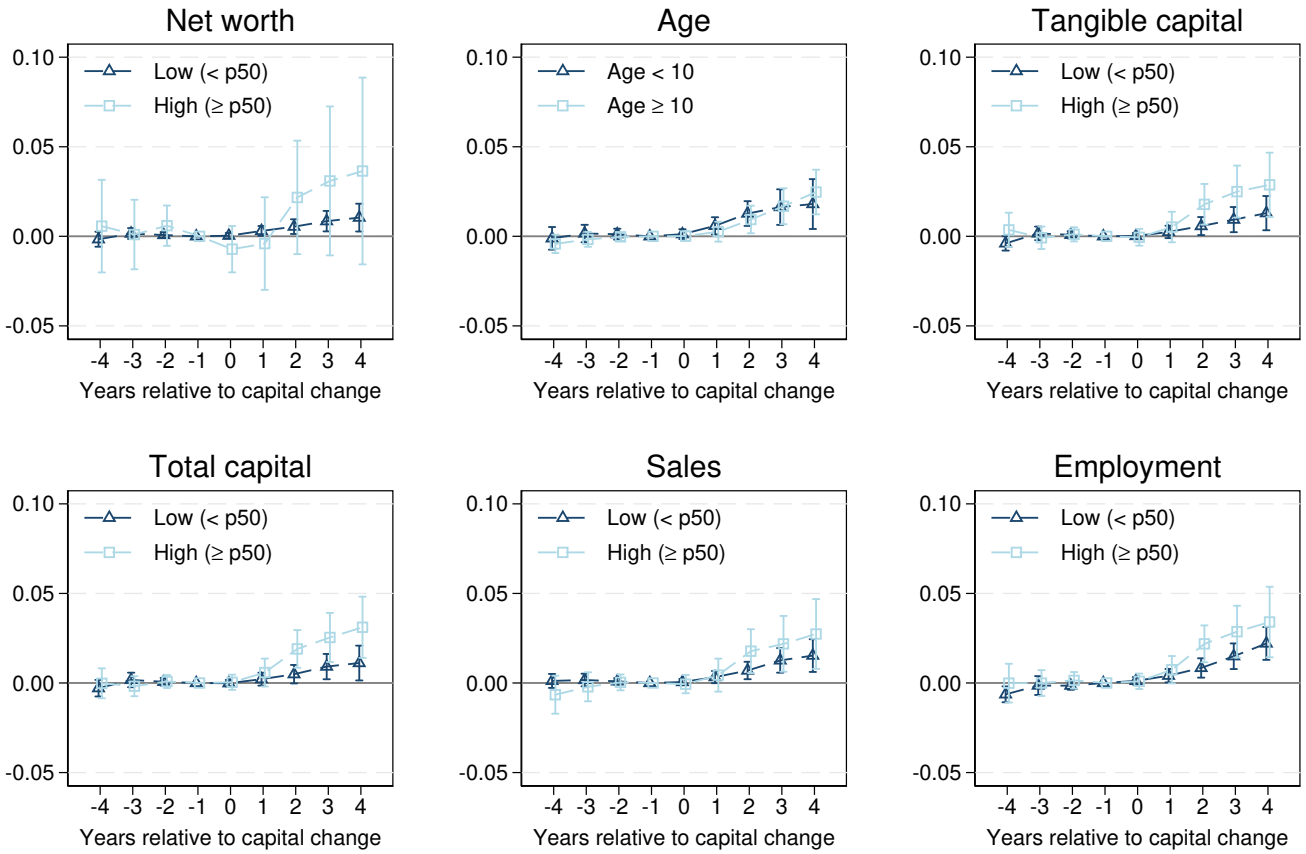
Notes: This figure shows the dynamic relationship between financial capital, investment, sales and patenting activity for the pool of European patenting firms. Each panel plots the estimates from equation (2) for the relevant dependent variable. Financial capital is the sum of debt, external equity and internal equity. Standard errors with 95% confidence intervals are clustered by firm.

Figure A.2: Financial capital, investment, sales and patenting: by source.



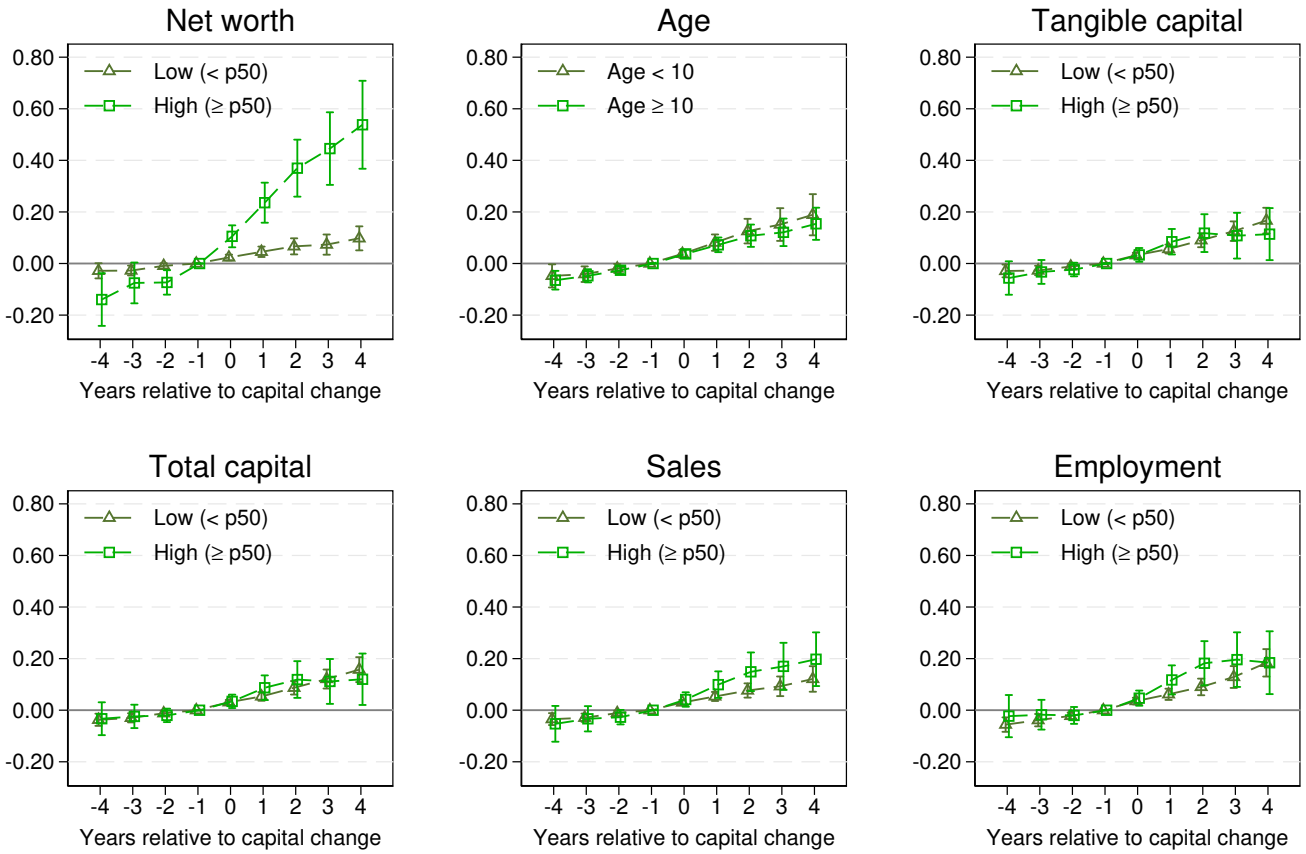
Notes: This figure shows the dynamic relationship between financial capital, investment, sales and patenting activity for the pool of European patenting firms. Each panel plots the estimates from equation (3) for the relevant dependent variable. Financial capital is the sum of debt, external equity and internal equity. Standard errors with 95% confidence intervals are clustered by firm.

Figure A.3: Internal equity and patenting, sample splits.



Notes: This figure shows the dynamic relationship between internal equity and patenting activity for the pool of European patenting firms. Each panel plots the estimates from equation (3), estimated separately on two subsamples partitioned by the relevant proxy for financial constraints. Standard errors with 95% confidence intervals are clustered by firm.

Figure A.4: Debt and patenting, sample splits.



Notes: This figure shows the dynamic relationship between debt and patenting activity for the pool of European patenting firms. Each panel plots the estimates from equation (3), estimated separately on two subsamples partitioned by the relevant proxy for financial constraints. Standard errors with 95% confidence intervals are clustered by firm.

B Theory Appendix

B.1 Static Pricing Equilibrium

There are $N+1$ producers in a given product line: one leader and $N \geq 2$ followers. All players seek to maximize profits. For clarity of exposition, assume the leader has access to the technology $y_L(k) = q_L k$, while followers produce using $y_i(k) = q_F k$ for all $i \in \{1, \ldots, N\}$, where $q_L > q_F$.[23] The leader can use at most $\bar{k} > 0$ units of capital, while followers do not face any limitation on capital usage. The final good producer will incur a total expenditure of Y for the product, so that $\sum_{i=1}^{N} p_i y_i + p_L y_L = Y$. All players engage in Bertrand competition, so only the cheapest variety of the good is bought. At equal prices, the leader is given priority; if the leader is not directly involved in the tie, any demand is equally split between the followers involved.[24] If the leader sets the lowest price but it cannot satisfy all demand, the remaining demand will go to the second cheapest producer(s). We find the Nash equilibrium of this game.

The case in which the leader can satisfy all demand pricing at the marginal cost of the followers is standard, so we focus on the case in which $mc_F q_L \bar{k} < Y$. First, we define $\bar{p}$ as the price at which the leader exactly satisfies all demand using the maximum amount of capital, $\bar{p} := \frac{Y}{q_L \bar{k}} > mc_F$. Given the game conditions, we can define the payoff functions of the leader and any follower as:

$$\pi_L(\boldsymbol{p}) = \begin{cases} (1 - \frac{r+\delta}{p_L q_L})Y & \text{if} \quad \bar{p} \leq p_L \leq p_{min,F}, \\ (p_L q_L - (r+\delta))\bar{k} & \text{if} \quad p_L \leq p_{min,F},\ p_L < \bar{p}, \\ 0 & \text{if} \quad p_L > p_{min,F}. \end{cases}$$

$$\pi_i(\boldsymbol{p}) = \begin{cases} \frac{1}{|p_{\text{tie}}|}(1 - \frac{r+\delta}{p_i q_F})Y & \text{if} \quad p_i < p_L,\ p_i = p_{min,F}, \\ \frac{1}{|p_{\text{tie}}|}(p_i - \frac{r+\delta}{q_F})RS & \text{if} \quad p_i \geq p_L,\ p_L < p_S,\ p_i = p_{min,F}, \\ 0 & \text{if} \quad (p_i \geq p_L \geq p_S,\ p_i = p_{min,F}) \ \vee \ (p_i > p_{min,F}), \end{cases}$$

where $\boldsymbol{p_F} := (p_1, \ldots, p_N)$, $\boldsymbol{p} := (p_L, \boldsymbol{p_F})$, $\boldsymbol{p_{F,-i}} := (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$, $p_{min,F} := \min\{\boldsymbol{p_F}\}$, $p_{min,F,-i} := \min\{\boldsymbol{p_{F,-i}}\}$, $|p_{\text{tie}}| := \sum_{i=1}^{N} \mathbf{1}\{p_i = p_{min,F}\}$, and $RS := Y - p_L q_L \bar{k}$.

It follows that the best response functions are:

$$p_L(\boldsymbol{p_F}) = \begin{cases} p_{min,F} & \text{if} \quad p_{min,F} > mc_L, \\ [p_{min,F}, \infty) & \text{if} \quad p_{min,F} = mc_L, \\ (p_{min,F}, \infty) & \text{if} \quad p_{min,F} < mc_L, \end{cases}$$

[23] The game can be easily extended to a Cobb-Douglas production function as in the main body of the paper with the same result.

[24] The resulting equilibrium does not depend on this assumption. The same result obtains if, under a tie between followers, all demand goes to any of them.

$$p_i(p_L, \boldsymbol{p_{F,-i}}) = \begin{cases} p_{min,F,-i} - \varepsilon & \text{if} \quad (mc_F < p_{min,F,-i} < p_L) \vee \\ & \quad \vee \ (mc_F < p_{min,F,-i},\ p_L \leq p_{min,F,-i},\ p_L \leq \bar{p}), \\ [p_{min,F,-i}, \infty) & \text{if} \quad (mc_F = p_{min,F,-i} < p_L) \vee \\ & \quad \vee \ (mc_F = p_{min,F,-i}\ p_L \leq p_{min,F,-i},\ p_L \leq \bar{p}), \\ (p_{min,F,-i}, \infty) & \text{if} \quad (mc_F > p_{min,F,-i},\ p_{min,F,-i} < p_L) \vee \\ & \quad \vee \ (mc_F > p_{min,F,-i},\ p_L \leq p_{min,F,-i},\ p_L \leq \bar{p}), \\ p_L - \varepsilon & \text{if} \quad p_{min,F,-i} \geq p_L > mc_F, \\ [p_L, \infty) & \text{if} \quad p_{min,F,-i} \geq p_L,\ p_L = mc_F, \\ (p_L, \infty) & \text{if} \quad p_{min,F,-i} \geq p_L,\ p_L < mc_F, \end{cases}$$

for an infinitesimally small $\varepsilon > 0$.[25] Note that the only fixed point and Nash equilibrium of the pricing game is $(p_L, \boldsymbol{p_F}) = (mc_F, \ldots, mc_F)$.

B.2 The Value Function

Consider an incumbent firm with net worth N_t and m product lines. Define the firm's problem in discrete time, where a step covers an infinitesimally small time length of Δt. With our assumptions, the value function of an incumbent firm solves the following equation:

$$\begin{aligned} V_t(N_t, m, \xi) = \max_{d \geq 0, z} \Bigg\{ & md Y_t \Delta t + e^{-r\Delta t} \Big[(zm\Delta t + o(\Delta t)) V_{t+\Delta t}(N_{t+\Delta t}, m+1, \xi) \\ & + (\tau m \Delta t + o(\Delta t)) V_{t+\Delta t}(N_{t+\Delta t}, m-1, \xi) \\ & + (\chi \Delta t + o(\Delta t)) V^s_{t+\Delta t}(N_{t+\Delta t}, m, \xi) \\ & + \mathbf{1}\{\xi = \xi_L\}(\iota_{L\to H}\Delta t + o(\Delta t)) V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_H) \\ & + \mathbf{1}\{\xi = \xi_H\}(\iota_{H\to L}\Delta t + o(\Delta t)) V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_L) \\ & + \Big(1 - zm\Delta t - \chi\Delta t - \tau m \Delta t - \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H}\Delta t \\ & - \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L}\Delta t + o(\Delta t)\Big) V_{t+\Delta t}(N_{t+\Delta t}, m, \xi) \Big] \Bigg\} \end{aligned} \tag{B.1}$$

$$\text{s.t.} \quad N_{t+\Delta t} = \Pi_t(N_t, m)\Delta t + (1 + r\Delta t)N_t - c^z_t(z)\Delta t - \left(md Y_t \Delta t + \frac{\kappa}{2} m d^2 Y_t \Delta t \right), \tag{B.2}$$

$$N_{t+\Delta t} \geq \underline{N}.$$

[25] Despite the best response being undefined for such small deviations, we frame the solution in this way for expositional purposes.

Subtract $e^{-r\Delta t}V_t(N_t, m, \xi)$ from both sides and divide by Δt in (B.1) and (B.2) to get:

$$\begin{aligned}\frac{1-e^{-r\Delta t}}{\Delta t}V_t(N_t,m,\xi) = \max_{d\geq 0, z}\Bigg\{ & md Y_t + e^{-r\Delta t}\Big[zm(V_{t+\Delta t}(N_{t+\Delta t}, m+1, \xi) - V_{t+\Delta t}(N_{t+\Delta t}, m, \xi)) \\ & + \tau m(V_{t+\Delta t}(N_{t+\Delta t}, m-1, \xi) - V_{t+\Delta t}(N_{t+\Delta t}, m, \xi)) \\ & + \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H}(V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_H) - V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_L)) \\ & + \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L}(V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_L) - V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_H)) \\ & + \chi(V^s_{t+\Delta t}(N_{t+\Delta t}, m, \xi) - V_{t+\Delta t}(N_{t+\Delta t}, m, \xi)) \\ & + \frac{V_{t+\Delta t}(N_{t+\Delta t}, m, \xi) - V_t(N_t, m, \xi)}{\Delta t} \\ & + \frac{o(\Delta t)}{\Delta t}(V_{t+\Delta t}(N_{t+\Delta t}, m+1, \xi) + V_{t+\Delta t}(N_{t+\Delta t}, m-1, \xi) \\ & + V^s_{t+\Delta t}(N_{t+\Delta t}, m, \xi) + \mathbf{1}\{\xi = \xi_L\}V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_H) \\ & + \mathbf{1}\{\xi = \xi_H\}V_{t+\Delta t}(N_{t+\Delta t}, m, \xi_L) + V_{t+\Delta t}(N_{t+\Delta t}, m, \xi))\Big]\Bigg\}\end{aligned}$$

$$\text{s.t.} \quad \frac{N_{t+\Delta t} - N_t}{\Delta t} = \Pi_t(N_t, m) + r_t N_t - c^z_t(z) - \left(dY_t + \frac{\kappa}{2}d^2 Y_t\right)m,$$
$$N_{t+\Delta t} \geq \underline{N},$$

where we have also rearranged some terms. Taking the limit as $\Delta t \to 0$, we obtain the expression:

$$\begin{aligned}rV_t(N_t,m,\xi) = \max_{d\geq 0, z}\Bigg\{ & md Y_t + zm(V_t(N_t, m+1, \xi) - V_t(N_t, m, \xi)) \\ & + \chi(V^s_t(N_t, m, \xi) - V_t(N_t, m, \xi)) \\ & + \tau m(V_t(N_t, m-1, \xi) - V_t(N_t, m, \xi)) \\ & + \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H}(V_t(N_t, m, \xi_H) - V_t(N_t, m, \xi_L)) \\ & + \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L}(V_t(N_t, m, \xi_L) - V_t(N_t, m, \xi_H)) \\ & + \underbrace{\lim_{\Delta t\to 0}\frac{V_{t+\Delta t}(N_{t+\Delta t}, m, \xi) - V_t(N_t, m, \xi)}{\Delta t}}_{=\frac{dV_t(N_t,m,\xi)}{dt} = \frac{\partial V_t(N_t,m,\xi)}{\partial t} + \frac{\partial V_t(N_t,m,\xi)}{\partial N_t}\frac{dN_t}{dt}}\Bigg\}\end{aligned}$$

$$\text{s.t.} \quad \frac{dN_t}{dt} = \Pi_t(N_t, m) + rN_t - c^z_t(z) - \left(dY_t + \frac{\kappa}{2}d^2 Y_t\right)m,$$
$$N_t \geq \underline{N}.$$

Last, rearranging terms, substituting the drift of net worth into the value function, and using Newton's notation for time derivatives, we obtain the following HJB equation:

$$
\begin{aligned}
rV_t(N_t,m,\xi) - \dot{V}_t(N_t,m,\xi) = \max_{d\geq 0,z} \Bigg\{ & md Y_t + \frac{\partial V_t(N_t,m,\xi)}{\partial N_t}\left(\Pi_t(N_t,m) + rN_t - c_t^z(z) - \left(d+\frac{\kappa}{2}d^2\right)mY_t\right) \\
& + zm(V_t(N_t,m+1,\xi) - V_t(N_t,m,\xi)) + \chi V_t^s(N_t,m,\xi) \\
& + \mathbf{1}\{\xi=\xi_L\}\iota_{L\to H}(V_t(N_t,m,\xi_H) - V_t(N_t,m,\xi_L)) \\
& + \mathbf{1}\{\xi=\xi_H\}\iota_{H\to L}(V_t(N_t,m,\xi_L) - V_t(N_t,m,\xi_H)) \\
& + \tau m(V_t(N_t,m-1,\xi) - V_t(N_t,m,\xi))\Bigg\} \qquad \text{(B.3)} \\
\text{s.t.} \quad & N_t \geq \underline{N}.
\end{aligned}
$$

We normalize the value function by final output, Y_t. Let $n_t := N_t/Y_t$. Then,

$$
\dot{n}_t = \frac{\dot{N}_t}{Y_t} - gn_t = \pi(n_t,m) + (r-g)n_t - \xi z^\psi m^\sigma - (d+\frac{\kappa}{2}d^2)m.
$$

Moreover, conjecture $V_t(N_t,m,\xi)/Y_t = v(n_t,m,\xi)$. This implies, under stationarity,

$$
\begin{aligned}
\frac{rV_t(N_t,m,\xi) - \dot{V}_t(N_t,m,\xi)}{Y_t} =& rv(n_t,m,\xi) - \frac{1}{Y_t}\frac{\partial(v(n_t,m,\xi)Y_t)}{\partial t} = \\
& rv(n_t,m,\xi) - \frac{1}{Y_t}\frac{\partial(Y_t)}{\partial t}v(n_t,m,\xi) + \frac{\partial(v(n_t,m,\xi))}{\partial n_t}\frac{N_t}{Y_t^2}\frac{\partial(Y_t)}{\partial t} = \\
& (r-g)v(n_t,m,\xi) + gn_t\frac{\partial v(n_t,m,\xi)}{\partial n_t},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial V_t(N_t,m,\xi)}{\partial N_t}\dot{N}_t &= \left(\frac{\partial v(n_t,m,\xi)}{\partial N_t}Y_t + \frac{\partial Y_t}{\partial N_t}v(n_t,m,\xi)\right)Y_t(\dot{n}_t + n_t g) \\
&= \frac{\partial v(n_t,m,\xi)}{\partial n_t}Y_t(\dot{n}_t + n_t g).
\end{aligned}
$$

Combining these results, substituting into the HJB equation, and using $\rho = r - g$, we get:

$$
\begin{aligned}
\rho v(n_t,m,\xi) = \max_{d\geq 0,z}\Bigg\{ & md + \frac{\partial v(n_t,m,\xi)}{\partial n_t}\left(\pi(n_t,m) + \rho n_t - \xi z^\psi m^\sigma - (d+\frac{\kappa}{2}d^2)m\right) \\
& + zm(v(n_t,m+1,\xi) - v(n_t,m,\xi)) + \chi v^s(n_t,m,\xi) \\
& + \mathbf{1}\{\xi=\xi_L\}\iota_{L\to H}(v(n_t,m,\xi_H) - v(n_t,m,\xi_L)) \\
& + \mathbf{1}\{\xi=\xi_H\}\iota_{H\to L}(v(n_t,m,\xi_L) - v(n_t,m,\xi_H))
\end{aligned}
$$

$$+\tau m(v(n_t, m-1, \xi) - v(n_t, m, \xi))\Big\} \tag{B.4}$$

$$\text{s.t.} \quad n_t \geq \underline{n}$$

where $\underline{n} = \underline{N}/Y_t$, and the expression only depends on normalized equity and it does not depend on time other than through n_t.

Note that, as in Achdou, Han, Lasry, Lions and Moll (2022), the borrowing constraint $n_t \geq \underline{n}$ never binds in the interior of the state space and as a result undistorted first order conditions hold everywhere. In other words, the first-order conditions to (B.4) hold with equality everywhere in the interior of the state space. Instead, the borrowing constraint gives rise to a state constraint boundary condition. Specifically, for the constraint not to be violated at $n_t = \underline{n}$, we have to ensure:

$$\pi(\underline{n}, m) + \rho\underline{n} - c^z(z(\underline{n}, m)) - (d(\underline{n}, m) + \frac{\kappa}{2}(d(\underline{n}, m))^2)m \geq 0.$$

B.3 The Stationary Distribution

In our model economy there is a continuum measure-one of product lines populated by firms who are heterogeneous in net worth, n_t, and number of product lines m. m evolves over time as a result of firms' innovations, while net worth evolves according to: $\dot{n}_t = s(n, m, \xi, t)$ where $s(\cdot)$ denotes the optimal "saving" policy function implied by optimal payout and innovation decisions.[26] Let $\phi(n, m, \xi, t)$ be the density of the joint distribution of n_t, m and ξ at time t, and $\Phi(n, m, \xi, t)$ be the corresponding CDF in the net worth dimension: $\Phi(n, m, \xi, t) = \Pr(n_t \leq n, m_t = m, \xi_t = \xi)$.

Consider the discrete-time analogue of the law of motion of net worth, over a time interval of length Δt:

$$n_t = n_{t+\Delta t} - s(n_{t+\Delta t}, m, \xi)\Delta t.$$

Momentarily ignoring changes in m and assuming firms decumulate resources, we have:

$$\Pr(n_{t+\Delta t} \leq n) = \Pr(n_t \leq n) + \Pr(n \leq n_t \leq n - s(n, m, \xi)\Delta t) = \Pr(n_t \leq n - s(n, m, \xi)\Delta t).$$

Next, taking into account changes in m, the fraction of firms with equity below n, number of products m, innovation productivity ξ, and entry, evolves as follows:

$$\Pr(n_{t+\Delta t} \leq n, m_{t+\Delta t} = m, \xi_{t+\Delta t} = \xi) =$$
$$\left[1 - \left(\frac{\int_{\underline{n}}^{n-s(n,m,\xi)\Delta t} mz(\nu, m, \xi)d\Phi(\nu, m, \xi, t)}{\Phi(n - s(n, m, \xi)\Delta t, m, \xi, t)} + \iota_{\xi\to\xi'} + m\tau\right)\Delta t\right] \cdot$$

[26] For notational simplicity, in what follows we suppress the dependence of the saving function on t.

$$
\begin{aligned}
&\cdot \Pr(n_t \leq n - s(n,m,\xi)\Delta t, m_t = m, \xi_t = \xi) - \int_{\mathcal{S}_t(n,m,\xi)} \chi \Delta t d\Phi(\nu, m, \xi, t) \\
&+ \frac{\int_{\underline{n}}^{n-s(n,m-1,\xi)\Delta t} (m-1) z(\nu, m-1, \xi) d\Phi(\nu, m-1, \xi, t)}{\Phi(n - s(n,m-1,\xi)\Delta t, m-1, \xi, t)} \Delta t \cdot \\
&\cdot \Pr(n_t \leq n - s(n,m-1,\xi)\Delta t, m_t = m-1, \xi_t = \xi)) \cdot \mathbf{1}\{m > 1\} \\
&+ \iota_{\xi' \to \xi} \Delta t \Pr(n_t \leq n - s(n,m,\xi')\Delta t, m_t = m, \xi_t = \xi') \\
&+ \tilde{\beta} e \Delta t \cdot P(n) \cdot \mathbf{1}\{m = 1\},
\end{aligned}
$$

where $\mathcal{S}_t(n,m,\xi)$ is the measure of type-ξ firms with m product lines that attain a resource level above n after matching with outside investors at time t but below it otherwise, $\tilde{\beta} := \beta \cdot \mathbf{1}\{\xi = L\} + (1-\beta) \cdot \mathbf{1}\{\xi = H\}$ is the probability of an entrant being of type ξ, and $P(n)$ is the cumulative density function over net worth at entry, or analogously

$$
\begin{aligned}
&\Phi(n,m,\xi,t+\Delta t) = \\
&\left[1 - \left(\frac{\int_{\underline{n}}^{n-s(n,m,\xi)\Delta t} m z(\nu,m,\xi) d\Phi(\nu,m,\xi,t)}{\Phi(n - s(n,m,\xi)\Delta t, m, \xi, t)} + \iota_{\xi \to \xi'} + m\tau \right) \Delta t \right] \Phi(n - s(n,m,\xi)\Delta t, m, \xi, t) \\
&- \int_{\mathcal{S}_t(n,m,\xi)} \chi \Delta t d\Phi(\nu, m, \xi, t) \\
&+ \frac{\int_{\underline{n}}^{n-s(n,m-1,\xi)\Delta t} (m-1) z(\nu, m-1, \xi) d\Phi(\nu, m-1, \xi, t)}{\Phi(n - s(n,m-1,\xi)\Delta t, m-1, \xi, t)} \Delta t \cdot \\
&\cdot \Phi(n - s(n,m-1,\xi)\Delta t, m-1, \xi, t) \cdot \mathbf{1}\{m > 1\} \\
&+ \iota_{\xi' \to \xi} \Delta t \Phi(n - s(n,m,\xi')\Delta t, m, \xi', t) \\
&+ \tilde{\beta} e \Delta t \cdot P(n) \cdot \mathbf{1}\{m = 1\}.
\end{aligned}
$$

Subtracting $\Phi(n,m,\xi,t)$ from both sides and dividing by Δt:

$$
\begin{aligned}
&\frac{\Phi(n,m,\xi,t+\Delta t) - \Phi(n,m,\xi,t)}{\Delta t} = \frac{\Phi(n - s(n,m,\xi)\Delta t, m, \xi, t) - \Phi(n,m,\xi,t)}{\Delta t} \\
&- \left(\frac{\int_{\underline{n}}^{n-s(n,m,\xi)\Delta t} m z(\nu,m,\xi) d\Phi(\nu,m,\xi,t)}{\Phi(n - s(n,m,\xi)\Delta t, m, \xi, t)} + \iota_{\xi \to \xi'} + m\tau \right) \Phi(n - s(n,m,\xi)\Delta t, m, \xi, t) \\
&- \int_{\mathcal{S}_t(n,m,\xi)} \chi d\Phi(\nu, m, \xi, t) \\
&+ \frac{\int_{\underline{n}}^{n-s(n,m-1,\xi)\Delta t} (m-1) z(\nu, m-1, \xi) d\Phi(\nu, m-1, \xi, t)}{\Phi(n - s(n,m-1,\xi)\Delta t, m-1, \xi, t)} \cdot \\
&\cdot \Phi(n - s(n,m-1,\xi)\Delta t, m-1, \xi, t) \cdot \mathbf{1}\{m > 1\} \\
&+ \iota_{\xi' \to \xi} \Phi(n - s(n,m,\xi')\Delta t, m, \xi', t) \\
&+ \tilde{\beta} e \cdot P(n) \cdot \mathbf{1}\{m = 1\},
\end{aligned}
$$

Taking the limit as $\Delta t \to 0$, and using

$$\lim_{\Delta t \to 0} \frac{\Phi(n - s(n,m,\xi)\Delta t, m, \xi, t) - \Phi(n,m,\xi,t)}{\Delta t}$$
$$\equiv \lim_{x \to 0} \frac{\Phi(n-x,m,\xi,t) - \Phi(n,m,\xi,t)}{x} s(n,m,\xi) = -s(n,m,\xi)\phi(n,m,\xi,t),$$

we get the Kolmogorov Forward (KF) equation:

$$\begin{aligned}
\dot{\Phi}(n,m,\xi,t) = &-s(n,m,\xi)\phi(n,m,\xi,t) - m\int_{\underline{n}}^{n} z(\nu,m,\xi)d\Phi(\nu,m,\xi,t) \\
&- \iota_{\xi\to\xi'}\Phi(n,m,\xi,t) - m\tau\Phi(n,m,\xi,t) - \int_{\mathcal{S}_t(n,m,\xi)} \chi d\Phi(\nu,m,\xi,t) \\
&+ \left((m-1)\int_{\underline{n}}^{n} z(\nu,m-1,\xi)d\Phi(\nu,m-1,\xi,t)\right)\mathbf{1}\{m>1\} \\
&+ \iota_{\xi'\to\xi}\Phi(n,m,\xi',t) + \tilde{\beta}e\cdot P(n)\cdot\mathbf{1}\{m=1\}. \qquad \text{(B.5)}
\end{aligned}$$

The first term of equation (B.5) captures inflows and outflows due to continuous movements in net worth. If $s(n,m,\xi) < 0$, the fraction of type-ξ firms with net worth below n and m product lines, $\Phi(n,m,\xi,t)$, increases at a rate proportional to the density $\phi(n,m,\xi,t) = \frac{\partial\Phi(n,m,\xi,t)}{\partial n}$; the reverse logic applies if $s(n,m,\xi) > 0$. The second, third and fourth terms adjusts for outflows due to innovation, innovation productivity type changes and displacement effects, respectively. The fifth one includes outflows due to equity injections by outside investors, the sixth one captures inflows from innovation by firms owning $m-1$ product lines, the seventh term represents inflows from type transitions, and the eighth one accounts for inflows due to entrants displacing incumbents.

To write the KF equation in terms of the density, we (weakly) differentiate (B.5) with respect to n to get:

$$\begin{aligned}
\dot{\phi}(n,m,\xi,t) = &\frac{\partial}{\partial n}\left[-s(n,m,\xi)\phi(n,m,\xi,t) - \int_{\mathcal{S}_t(n,m,\xi)} \chi d\Phi(\nu,m,\xi,t)\right] \\
&- (mz(n,m,\xi) + \iota_{\xi\to\xi'} + m\tau)\phi(n,m,\xi,t) \\
&+ (m-1)z(n,m-1,\xi)\phi(n,m-1,\xi,t)\cdot\mathbf{1}\{m>1\} \\
&+ \iota_{\xi'\to\xi}\phi(n,m,\xi',t) + \tilde{\beta}e\cdot\varrho(n)\cdot\mathbf{1}\{m=1\}, \qquad \text{(B.6)}
\end{aligned}$$

where $\varrho(n)$ is the probability density function over net worth at entry. For $T^*(n,m,\xi)$ monotonically decreasing in net worth, the measure $\mathcal{S}_t(n,m,\xi)$ is a convex set in net worth with upper bound $\overline{n}_p(n,m,\xi) = n$. Denoting by $\underline{n}_p(n,m,\xi)$ its lower bound, the term $\frac{\partial}{\partial n}\int_{\mathcal{S}_t(n,m,\xi)} \chi\, d\Phi(\nu,m,\xi,t)$ becomes

$$\chi\,\phi(n,m,\xi,t) - \frac{\partial \underline{n}_p(n,m,\xi)}{\partial n}\chi\,\phi(\underline{n}_p(n,m,\xi),m,\xi,t).$$

Last, the stationary equilibrium satisfies:

$$\dot{\phi}(n,m,\xi,t) = 0, \forall m \in \{1,\dots,\overline{m}\}, \forall \xi \in \{\xi_L, \xi_H\},$$
$$\sum_{\xi \in \{\xi_L,\xi_H\}} \sum_{m=1}^{\overline{m}} m \int_{\underline{n}}^{\infty} \phi(n,m,\xi,t) dn = 1.$$

B.4 The Growth Rate

To derive the growth rate of the productivity frontier, note that, for an infinitesimally small time interval Δt, we have:

$$\ln Q_{t+\Delta t} = \int_0^1 \ln(q_{j,t+\Delta t}) dj.$$

Defining $\mathcal{J}_t \subset [0,1]$ as the set of product lines that experience an innovation between t and $t+\Delta t$, we can partition the above integral as:

$$\ln Q_{t+\Delta t} = \int_{j\in\mathcal{J}_t} \ln(\lambda q_{j,t}) dj + \int_{j\notin\mathcal{J}_t} \ln(q_{j,t}) dj = \ln Q_t + \int_{j\in\mathcal{J}_t} \ln(\lambda) dj. \tag{B.7}$$

Each incumbent firm with m product lines generates innovations at rate zm, each of which lands on a uniformly drawn product line, and potential entrants generate innovations at rate e. Letting z_{jt} denote the innovation intensity of the firm that owns product line j, the measure of product lines that experience an innovation between t and $t+\Delta t$ is therefore $\big(\int_0^1 z_{jt}\, dj + e\big)\Delta t + o(\Delta t)$, while all other lines maintain the same productivity. Consequently, we can manipulate equation (B.7) to obtain:

$$\frac{\ln(Q_{t+\Delta t}/Q_t)}{\Delta t} = \ln(\lambda)\left(\int_0^1 z_{jt}\, dj + e\right) + \frac{o(\Delta t)}{\Delta t}.$$

Last, taking the limit $\Delta t \to 0$, we find the growth rate of the productivity frontier:

$$g = \ln(\lambda)\left(\int_0^1 z_{jt}\, dj + e\right). \tag{B.8}$$

B.5 Derivation of the Aggregate Resource Constraint

Aggregating the drift (10) over the stationary distribution of incumbents and adding the flows due to equity injections and entry, we obtain:

$$\dot{\mathcal{N}}_t = \Pi_t^{agg} + r_t \mathcal{N}_t - C_t^z - C_t^d + \chi \int T_t Y_t\, d\phi_t + e\, \mathbb{E}[n^0]\, Y_t, \tag{B.9}$$

where $\mathcal{N}_t \equiv \int_{\mathcal{F}} N_{ft}\, df$, $\Pi_t^{agg} = \mathcal{R}_t^I - w_t L_t^I - R_t K_t^I$ is the aggregate production profit, C_t^z and C_t^d are aggregate R&D and dividend payout costs.

The capital operated by the fringe of followers, K_t^F, is owned directly by the representative household and rented at the competitive rate $R_t = r_t + \delta$. Due to standard price cutting arguments, followers earn zero profits:

$$\mathcal{R}_t^F = w_t L_t^F + R_t K_t^F, \tag{B.10}$$

where L_t^F is total labor used by followers.

The household holds firm equity $\mathcal{V}_t \equiv \int_{\mathcal{F}} V_{ft}\, df$, firm debt B_t, and the capital rented to followers K_t^F, so that $A_t = \mathcal{V}_t + B_t + K_t^F$. The dynamics of the aggregate equity position are pinned down by no-arbitrage between equity and the riskless asset. The cash flows accruing to the equity claim per unit time consist of the aggregate dividend received, D_t, minus the equity injections funded through the financial intermediary, $\chi \int (T_t + \zeta T_t^2/2) Y_t\, d\phi_t$, minus the net worth provided to entering firms, $e\, \mathbb{E}[n^0] Y_t$.

$$\dot{\mathcal{V}}_t = r_t \mathcal{V}_t - D_t + \chi \int (T_t + \zeta T_t^2/2) Y_t\, d\phi_t + e\, \mathbb{E}[n^0]\, Y_t. \tag{B.11}$$

Substituting (B.11) and the asset decomposition into the household budget $\dot{A}_t = r_t A_t + w_t - C_t - C_t^e$, and collecting flows,

$$\dot{B}_t + \dot{K}_t^F + \delta K_t^F = r_t B_t + R_t K_t^F + w_t + D_t - \chi \int (T_t + \zeta T_t^2/2) Y_t\, d\phi_t - e\, \mathbb{E}[n^0]\, Y_t - C_t - C_t^e. \tag{B.12}$$

Substituting $\mathcal{N}_t = K_t^I - B_t$ into (B.9), we find an expression for incumbent investment $\dot{K}_t^I + \delta K_t^I$. Adding it to (B.12), the debt terms $\dot{B}_t$ and $r_t B_t$, the equity injections $\chi \int T_t Y_t\, d\phi_t$, and the entry net worth $e\, \mathbb{E}[n^0] Y_t$ cancel out. Substituting the fringe relation (B.10) and applying labor market clearing $L_t^I + L_t^F = 1$ eliminates the wage bill, $w_t L_t^I + w_t L_t^F = w_t$, and replaces $R_t K_t^F$ with $\mathcal{R}_t^F - w_t L_t^F$. The revenue identity $\mathcal{R}_t^I + \mathcal{R}_t^F = Y_t$ then collapses the remaining revenue and labor terms into Y_t:

$$\dot{K}_t^I + \delta K_t^I + \dot{K}_t^F + \delta K_t^F = Y_t - C_t^z - C_t^d + D_t - C_t^e - C_t - \chi \frac{\zeta}{2} Y_t \int T_t^2\, d\phi_t.$$

Combining incumbent and fringe investment into aggregate gross investment $\dot{K}_t + \delta K_t$ with $K_t \equiv K_t^I + K_t^F$, and imposing the balanced growth path condition $\dot{K}_t = g K_t$, gives

$$1 = \frac{C_t}{Y_t} + (g + \delta) \frac{K_t}{Y_t} + \frac{C_t^z}{Y_t} + \frac{C_t^e}{Y_t} + \frac{\kappa}{2} \int d_{jt}^2\, dj + \chi \frac{\zeta}{2} \int T_t^2\, d\phi_t.$$

B.6 Initial Aggregate Output

Computing welfare through equation (37) requires the initial level of consumption C_0, which, from equation (38), in turn requires the initial level of output Y_0. We pin down Y_0 as follows. We first impose the initial condition $Q_0 = 1$, which through equation (32) recovers the initial wage w_0. Given w_0, the level of output is determined by the labor market clearing condition $\int_0^1 (l_{ijt} + l_{-ijt})\, dj = 1$.

To exploit it, note that all optimal labor inputs are proportional to output. The constrained inputs depend on $\bar{k}_t(N_t, m_t)$, which is itself linear in output: revenues per product line are $p_{jt}y_{jt} = Y_t$ when the leader serves the entire demand, and $p_{jt}y_{jt} = R_t\, \mu^{\frac{1}{\alpha}} k_{jt}/\alpha$ when it produces with l_{jt}^C. Substituting into the collateral constraint,

$$\bar{k}_t(N_t, m_t) = Y_t\, \widetilde{k}(n_t/m_t),$$

where

$$\widetilde{k}(n_t/m_t) \equiv \min\left\{\theta_n \frac{n_t}{m_t} + \theta_y,\ \frac{\theta_n\, n_t/m_t}{1 - \theta_y R_t\, \mu^{\frac{1}{\alpha}}/\alpha}\right\}$$

is the normalized capital limit and $n_t \equiv N_t/Y_t$. The labor-to-output ratios are therefore independent of the level of output:

$$\begin{aligned}
\frac{l_{jt}^*}{Y_t} &= \frac{1-\alpha}{w_t}\frac{1}{\mu}, \\
\frac{l_{jt}^C}{Y_t} &= \frac{1-\alpha}{\alpha}\frac{R_t}{w_t}\, \mu^{\frac{1}{\alpha}}\, \widetilde{k}(n_t/m_t), \\
\frac{l_{jt}^D}{Y_t} &= \frac{1-\alpha}{w_t}\left[\left(\frac{\alpha}{R_t\, \widetilde{k}(n_t/m_t)}\right)^{\alpha}\frac{1}{\mu}\right]^{\frac{1}{1-\alpha}}.
\end{aligned}$$

Each constrained line uses $\min\{l_{jt}^C, l_{jt}^D\}$, and the branch it falls on depends only on n_t/m_t, hence not on the level of output. Defining the aggregate labor-to-output ratio $\Lambda_t \equiv \int_0^1 (l_{ijt} + l_{-ijt})/Y_t\ dj$, which is constant, labor market clearing yields $Y_0 = 1/\Lambda_0$. The remaining elements of equation (38) are already known from the stationary equilibrium.

C Numerical Appendix

We solve for the BGP equilibrium of the model as follows. In an outer loop, we solve a fixed point algorithm on creative destruction, τ. In an inner loop, we solve for firms' value functions and calculate their optimal R&D and dividend payout decisions. In other words, we get the solutions to the firms' HJB equations conditional on our creative destruction guess in the outer loop. Having solved optimal innovation policies, we calculate the implied creative destruction rate and update the outer loop as

appropriate. Then, deriving the KF equation, we solve for the steady state firm distribution.

C.1 HJB Equation

We discretize and solve HJB equation (21) numerically using a finite difference method (Achdou *et al.*, 2022), which allows us to write it as a system of non-linear equations.

For each $m \in \{1, 2, \dots, \overline{m}\}$ and $\xi \in \{\xi_L, \xi_H\}$, we approximate v at I discrete points, $n_i, i = 1, \dots, I$, using an equispaced grid with distance Δn between points. We use the short-hand notation $v_{i,m,\xi} = v(n_i, m, \xi)$. There are different possibilities to approximate the partial derivative $\partial v(n_i, m, \xi)/\partial n$ at a given grid point n_i. The forward and backward approximations to $\partial v(n_i, m, \xi)/\partial n$ are respectively:

$$\frac{\partial v(n_i, m, \xi)}{\partial n} \approx \frac{v_{i+1,m,\xi} - v_{i,m,\xi}}{\Delta n} =: \left(\frac{\partial v}{\partial n}\right)_{i,m,\xi,F}$$

and

$$\frac{\partial v(n_i, m, \xi)}{\partial n} \approx \frac{v_{i,m,\xi} - v_{i-1,m,\xi}}{\Delta n} =: \left(\frac{\partial v}{\partial n}\right)_{i,m,\xi,B}.$$

First, we solve for T^* from equation (26). Since the equation is nonlinear, we must find this value numerically.[27] Then, we can compute $\Omega^*_{i,m,\xi} = v_{j(i,m,\xi),m,\xi} - v_{i,m,\xi} - c_f(T^*_{i,m,\xi})$, where $j = j(i, m, \xi)$ is a mapping from net worth, product lines and innovation productivity type, (i, m, ξ), to the grid point that is closest to the equity level $n_i + T^*_{i,m,\xi}$.[28] The finite difference approximation to (21) is then:

$$\begin{aligned}
\rho v_{i,m,\xi} = {} & m d_{i,m,\xi} + \left(\frac{\partial v}{\partial n}\right)_{i,m,\xi} \underbrace{\left(\pi(n_i, m) + \rho n_i - c^z(z_{i,m,\xi}) - \left(d_{i,m,\xi} + \frac{\kappa}{2} d^2_{i,m,\xi}\right) m\right)}_{=: s_{i,m,\xi}} \\
& + z_{i,m,\xi} m (v_{i,m+1,\xi} - v_{i,m,\xi}) + \mathbf{1}\{\xi = \xi_L\} \iota_{L \to H} (v_{i,m,\xi_H} - v_{i,m,\xi_L}) \\
& + \mathbf{1}\{\xi = \xi_H\} \iota_{H \to L} (v_{i,m,\xi_L} - v_{i,m,\xi_H}) \\
& + \chi \eta \Omega^*_{i,m,\xi} + \tau m (v_{i,m-1,\xi} - v_{i,m,\xi}),
\end{aligned} \tag{C.1}$$

for $i = 1, \dots, I, m = 1, \dots, \overline{m}$, where

$$d_{i,m,\xi} = \frac{1}{\kappa}\left((1 + \Xi)\left(\frac{\partial v}{\partial n}\right)^{-1}_{i,m,\xi} - 1\right), \quad z_{i,m,\xi} = (c^{z\prime})^{-1}\left(m \frac{v_{i,m+1,\xi} - v_{i,m,\xi}}{(\partial v/\partial n)_{i,m,\xi}}\right),$$

and $(\partial v/\partial n)_{i,m,\xi}$ is either the forward or backward difference approximation.

We use the so-called upwind scheme, which involves using a forward difference approximation whenever the drift of net worth is positive and backward difference whenever the drift of net worth is

[27] Since a closed-form expression for $v_{i,m,\xi}$ does not exist, we must interpolate across the grid for $v_{i,m,\xi}$ and call a non-linear solver.

[28] In practice, $v_{j,m,\xi}$ is the interpolated value function at $n_j = n_i + T^*_{i,m,\xi}$.

negative. In practice, this is done as follows. First, we compute the drift according to both the forward and backwards difference approximations:

$$s_{i,m,\xi,F} = \pi(n_i, m) + \rho n_i - c^z(z_{i,m,\xi,F}) - (d_{i,m,\xi,F} + \frac{\kappa}{2} d^2_{i,m,\xi,F})m$$

and

$$s_{i,m,\xi,B} = \pi(n_i, m) + \rho n_i - c^z(z_{i,m,\xi,B}) - (d_{i,m,\xi,B} + \frac{\kappa}{2} d^2_{i,m,\xi,B})m$$

where

$$d_{i,m,\xi,j} = \frac{1}{\kappa}\left((1+\Xi)\left(\frac{\partial v}{\partial n}\right)^{-1}_{i,m,\xi,j} - 1\right), \quad z_{i,m,\xi,j} = (c^{z\prime})^{-1}\left(m\frac{v_{i,m+1,\xi} - v_{i,m,\xi}}{(\partial v/\partial n)_{i,m,\xi,j}}\right),$$

for $j = F, B$. Then, if $s_{i,m,\xi,F} > 0$, we use the forward difference to approximate the partial derivative, while if $s_{i,m,\xi,B} < 0$ we use the backwards difference. If instead $s_{i,m,\xi,F} \leq 0 \leq s_{i,m,\xi,B}$, we set the drift equal to 0.[29] Therefore, letting $s^+_{i,m,\xi,F} = \max\{s_{i,m,\xi,F}, 0\}$ and $s^-_{i,m,\xi,B} = \min\{s_{i,m,\xi,B}, 0\}$, the upwind version of (C.1) is:

$$\begin{aligned}\rho v_{i,m,\xi} = {} & m d_{i,m,\xi} + \frac{v_{i+1,m,\xi} - v_{i,m,\xi}}{\Delta n} s^+_{i,m,\xi,F} + \frac{v_{i,m,\xi} - v_{i-1,m,\xi}}{\Delta n} s^-_{i,m,\xi,B} + \\ & + z_{i,m,\xi} m (v_{i,m+1,\xi} - v_{i,m,\xi}) + \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H}(v_{i,m,\xi_H} - v_{i,m,\xi_L}) \\ & + \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L}(v_{i,m,\xi_L} - v_{i,m,\xi_H}) + \chi\eta\Omega^*_{i,m,\xi} + \tau m (v_{i,m-1,\xi} - v_{i,m,\xi}),\end{aligned} \tag{C.2}$$

for $i = 1, \ldots, I, m = 1, \ldots, \overline{m}, \xi = \xi_L, \xi_H$, where $d_{i,m,\xi}$ and $z_{i,m,\xi}$ use an FD approximation of $(\partial v(n_i, m, \xi)/\partial n)$ that depends on the sign of $s_{i,m,\xi}$ in the same way.

This upwind scheme handles the state constraint boundary condition easily. The state constraint is enforced by setting $s_{1,m,\xi,B} = 0$ and computing $(\partial v/\partial n)_{1,m,\xi,B}$ accordingly. Thus, at the lower end of the state space, if $s_{1,m,\xi,F} > 0$ we use the forward difference approximation, and the state constraint is only imposed when the forward difference approximation would result in $s_{1,m,\xi,F} \leq 0$. This ensures that the borrowing constraint is never violated. At the upper end of the state space, the upwind scheme makes sure that a backward difference approximation is used. Indeed, if n_I is sufficiently large, the drift is negative.

The FD approximation is highly non-linear, and hence it has to be solved using an iterative scheme. In particular, we use the following algorithm. Guess $v^0_{i,m,\xi}, i = 1, \ldots, I, m = 1, \ldots, \overline{m}, \xi = \xi_L, \xi_H$ and for $n = 0, 1, 2, \ldots$:

1. Compute $(\partial v^n/\partial n)_{i,m,\xi}$ using finite difference approximation and the upwind scheme.
2. Solve first order conditions given $(\partial v^n/\partial n)_{i,m,\xi}$, to derive $d^n_{i,m,\xi}, z^n_{i,m,\xi}$ and $s^n_{i,m,\xi}$.

[29] Note that the concavity of v implies $s_{i,m,\xi,F} < s_{i,m,\xi,B}$.

3. Update v^n using an "implicit method":

$$\begin{aligned}\frac{v^{n+1}_{i,m,\xi} - v^{n}_{i,m,\xi}}{\Delta} + \rho v^{n+1}_{i,m,\xi} = m d^{n}_{i,m,\xi} + \frac{v^{n+1}_{i+1,m,\xi} - v^{n+1}_{i,m,\xi}}{\Delta n}(s^{n}_{i,m,\xi,F})^{+} + \frac{v^{n+1}_{i,m,\xi} - v^{n+1}_{i-1,m,\xi}}{\Delta n}(s^{n}_{i,m,\xi,B})^{-} \\ + z^{n}_{i,m,\xi} m (v^{n+1}_{i,m+1,\xi} - v^{n+1}_{i,m,\xi}) + \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H}(v^{n+1}_{i,m,\xi_H} - v^{n+1}_{i,m,\xi_L}) \\ + \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L}(v^{n+1}_{i,m,\xi_L} - v^{n+1}_{i,m,\xi_H}) + \chi\eta\Omega^{*}_{i,m,\xi} + \tau m (v^{n+1}_{i,m-1,\xi} - v^{n+1}_{i,m,\xi}).\end{aligned} \tag{C.3}$$

4. If v^{n+1} is sufficiently close to v^n, stop. Otherwise, go to Step 1.

Equation (C.3) constitutes a system of $I \times \overline{m} \times 2$ linear equations and can be written in matrix notation as follows. First, rewrite (C.3) as:

$$\begin{aligned}\frac{v^{n+1}_{i,m,\xi} - v^{n}_{i,m,\xi}}{\Delta} + \rho v^{n+1}_{i,m,\xi} =& m d^{n}_{i,m,\xi} - \chi\eta c_f(T^{*}_{i,m,\xi}) + v^{n+1}_{i-1,m,\xi} x_{i,m,\xi} + v^{n+1}_{i,m,\xi} y_{i,m,\xi} + v^{n+1}_{i+1,m,\xi} w_{i,m,\xi} \\ &+ z^{n}_{i,m,\xi} m v^{n+1}_{i,m+1,\xi} + \eta\chi {v_{j,m,\xi}}^{n+1} + \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H} v^{n+1}_{i,m,\xi_H} \\ &+ \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L} v^{n+1}_{i,m,\xi_L} + \tau m v^{n+1}_{i,m-1,\xi}\end{aligned} \tag{C.4}$$

where[30]

$$x_{i,m,\xi} = -\frac{(s^{n}_{i,m,\xi,B})^{-}}{\Delta n},$$

$$y_{i,m,\xi} = -\frac{(s^{n}_{i,m,\xi,F})^{+}}{\Delta n} + \frac{(s^{n}_{i,m,\xi,B})^{-}}{\Delta n} - z^{n}_{i,m,\xi} m - \eta\chi - \mathbf{1}\{\xi = \xi_L\}\iota_{L\to H} - \mathbf{1}\{\xi = \xi_H\}\iota_{H\to L} - \tau m,$$

and $w_{i,m,\xi} = \dfrac{(s^{n}_{i,m,\xi,F})^{+}}{\Delta n}$.

This system of equations can be written in matrix notation as:

$$\frac{1}{\Delta}(v^{n+1} - v^{n}) + \rho v^{n+1} = d^{n} + \mathbf{A}^{n} v^{n+1}, \tag{C.5}$$

[30] Note that $x_{1,m,\xi} = w_{I,m,\xi} = 0, \forall m, \xi$, so $v^{n+1}_{0,m,\xi}$ and $v^{n+1}_{I+1,m,\xi}$ are never used.

where

$$\mathbf{A}^n = \begin{bmatrix}
y_{1,1,\xi_L} & w_{1,1,\xi_L} & 0 & \dots & \dots & 0 & mz^n_{1,1,\xi_L} & 0 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
x_{2,1,\xi_L} & y_{2,1,\xi_L} & w_{2,1,\xi_L} & 0 & \dots & \vdots & 0 & mz^n_{2,1,\xi_L} & 0 & 0 & \dots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & x_{3,1,\xi_L} & y_{3,1,\xi_L} & w_{3,1,\xi_L} & 0 & \vdots & 0 & 0 & mz^n_{3,1,\xi_L} & 0 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & 0 & x_{I,1,\xi_L} & y_{I,1,\xi_L} & 0 & \dots & \dots & \dots & \ddots & mz^n_{I,1,\xi_L} & 0 & \dots & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 & y_{1,\overline{m},\xi_H} & w_{1,\overline{m},\xi_H} & 0 & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & x_{2,\overline{m},\xi_H} & y_{2,\overline{m},\xi_H} & w_{2,\overline{m},\xi_H} & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & x_{3,\overline{m},\xi_H} & y_{3,\overline{m},\xi_H} & w_{3,\overline{m},\xi_H} & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & x_{I,\overline{m},\xi_H} & y_{I,\overline{m},\xi_H}
\end{bmatrix}$$

and

$$d^n = \begin{bmatrix}
md^n_{1,1,\xi_L} - \chi\eta c_f(T^*_{1,1,\xi_L}) \\
\vdots \\
md^n_{I,1,\xi_L} - \chi\eta c_f(T^*_{I,1,\xi_L}) \\
\vdots \\
\vdots \\
md^n_{1,\overline{m},\xi_L} - \chi\eta c_f(T^*_{1,\overline{m},\xi_L}) \\
\vdots \\
md^n_{I,\overline{m},\xi_L} - \chi\eta c_f(T^*_{I,\overline{m},\xi_L}) \\
md^n_{1,1,\xi_H} - \chi\eta c_f(T^*_{1,1,\xi_H}) \\
\vdots \\
md^n_{I,1,\xi_H} - \chi\eta c_f(T^*_{I,1,\xi_H}) \\
\vdots \\
\vdots \\
md^n_{1,\overline{m},\xi_H} - \chi\eta c_f(T^*_{1,\overline{m},\xi_H}) \\
\vdots \\
md^n_{I,\overline{m},\xi_H} - \chi\eta c_f(T^*_{I,\overline{m},\xi_H})
\end{bmatrix}$$

The $2I\overline{m} \times 2I\overline{m}$ matrix $\mathbf{A}^n$ has the following properties: all its diagonal entries are non-positive; all off-diagonal entries are non-negative; all rows sum to zero.[31]

Analogously, we can write this system as:

$$\mathbf{B}^n v^{n+1} = b^n, \tag{C.6}$$

[31] If we consider the case with an infinite updating step size, $\Delta \to \infty$, so that $1/\Delta = 0$, our linear system reduces to: $\rho v^{n+1} = d^n + \mathbf{A}^n v^{n+1}$, which is just a way of writing the discretized version of the HJB equation in matrix form.

where

$$\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n \quad \text{and} \quad b^n = d^n + \frac{1}{\Delta}v^n.$$

Equation (C.6) can be solved efficiently using sparse matrix routines.

Note that when solving the backward differential at $n_1 = \underline{n}$, we must satisfy:

$$\frac{m\left(v_{1,m+1,\xi} - v_{1,m,\xi}\right)}{c^{z\prime}(z_{1,m,\xi,B})} = \frac{1+\Xi}{1+\kappa d_{1,m,\xi,B}}, \tag{C.7}$$

$$\pi(n_1,m) + \rho n_1 = c^z(z_{1,m,\xi,B}) + \left(d_{1,m,\xi,B} + \frac{\kappa}{2}(d_{1,m,\xi,B})^2\right)m. \tag{C.8}$$

That is, dividend payouts and R&D intensity must equalize the shadow value of funds across first order conditions, and the drift of net worth has to be zero. This reduces to a system of two non-linear equations in two unknowns. Once we solve for the policy functions, we obtain a new guess for the derivative of the value function with respect to net worth.

C.2 Kolmogorov Forward Equation

In a stationary equilibrium, the KF equations describing the evolution of the joint distribution of number of product lines and net worth must satisfy:

$$\begin{aligned}
0 &= \frac{\partial}{\partial n}\left[-s(n,1,\xi)\phi(n,1,\xi)\right] - \chi\phi(n,1,\xi) - \frac{\partial \underline{n}_p(n,1,\xi)}{\partial n}\chi\phi(\underline{n}_p(n,1,\xi),1,\xi) \\
&\quad - (z(n,1,\xi) + \iota_{\xi\to\xi'} + \tau)\phi(n,1,\xi) + \iota_{\xi'\to\xi}\phi(n,1,\xi') + \tilde{\beta}e\varrho(n), \\
0 &= \frac{\partial}{\partial n}\left[-s(n,m,\xi)\phi(n,m,\xi)\right] - \chi\phi(n,m,\xi) - \frac{\partial \underline{n}_p(n,m,\xi)}{\partial n}\chi\phi(\underline{n}_p(n,m,\xi),m,\xi) \\
&\quad - (mz(n,m,\xi) + \iota_{\xi\to\xi'} + m\tau)\phi(n,m,\xi) + \iota_{\xi'\to\xi}\phi(n,m,\xi') \\
&\quad + (m-1)z(n,m-1,\xi)\phi(n,m-1,\xi), \quad \text{for} \quad m = 2,\ldots,\overline{m}, \\
1 &= \sum_{\xi\in\{\xi_L,\xi_H\}}\sum_{m=1}^{\overline{m}} m\int_{\underline{n}}^{\infty}\phi(n,m,\xi)dn.
\end{aligned}$$

We can discretize the above conditions as

$$\begin{aligned}
0 &= -\left[s_{i,1,\xi}\phi_{i,1,\xi}\right]' - \chi\phi_{i,1,\xi} - \cdots - (z_{i,1,\xi} + \iota_{\xi\to\xi'} + \tau)\phi_{i,1,\xi} + \iota_{\xi'\to\xi}\phi_{i,1,\xi'} + \tilde{\beta}e\varrho(i), \\
0 &= -\left[s_{i,m,\xi}\phi_{i,m,\xi}\right]' - \chi\phi_{i,m,\xi} - \cdots - (mz_{i,m,\xi} + \iota_{\xi\to\xi'} + m\tau)\phi_{i,m,\xi} + \iota_{\xi'\to\xi}\phi_{i,m,\xi'} \\
&\quad + (m-1)z_{i,m-1,\xi}\phi_{i,m-1,\xi}, \quad \text{for } m = 2,\ldots,\overline{m}, \\
1 &= \sum_{\xi\in\{\xi_L,\xi_H\}}\sum_{m=1}^{\overline{m}} m\sum_{i=1}^{I}\phi_{i,m,\xi}\Delta n.
\end{aligned}$$

Notice that the KF equations are linear in $\phi(\cdot)$ and hence so will be their finite difference approximation.

As a result, no iterative procedure is needed and the equation can be solved in one step.

We again discretize these equations using a first-difference scheme. In particular, we propose the following approximation, which resembles the one used to build $\mathbf{A}$ in Section C.1:

$$\begin{aligned}
0 = &- \frac{(s^n_{i,1,\xi,F})^+\phi_{i,1,\xi} - (s^n_{i-1,1,\xi,F})^+\phi_{i-1,1,\xi}}{\Delta n} - \frac{(s^n_{i+1,1,\xi,B})^-\phi_{i+1,1,\xi} - (s^n_{i,1,\xi,B})^-\phi_{i,1,\xi}}{\Delta n} \\
&- \chi\phi_{i,1,\xi} - \cdots - (z_{i,1,\xi} + \iota_{\xi\to\xi'} + \tau)\phi_{i,1,\xi} + \iota_{\xi'\to\xi}\phi_{i,1,\xi'} + \tilde{\beta}e\varrho(i), \\
0 = &- \frac{(s^n_{i,m,\xi,F})^+\phi_{i,m,\xi} - (s^n_{i-1,m,\xi,F})^+\phi_{i-1,m,\xi}}{\Delta n} - \frac{(s^n_{i+1,m,\xi,B})^-\phi_{i+1,m,\xi} - (s^n_{i,m,\xi,B})^-\phi_{i,m,\xi}}{\Delta n} \\
&- \chi\phi_{i,m,\xi} - \cdots - (mz_{i,m,\xi} + \iota_{\xi\to\xi'} + m\tau)\phi_{i,m,\xi} + \iota_{\xi'\to\xi}\phi_{i,m,\xi'} + (m-1)z_{i,m-1,\xi}\phi_{i,m-1,\xi} \\
&\text{for} \quad m = 2,\dots,\overline{m}, \xi \in \{\xi_L, \xi_H\}.
\end{aligned}$$

Recycling previous notation we can write this more compactly as follows:

$$\begin{aligned}
0 &= x_{i+1,1,\xi}\phi_{i+1,1,\xi} + y_{i,1,\xi}\phi_{i,1,\xi} + w_{i-1,1,\xi}\phi_{i-1,1,\xi} + \iota_{\xi'\to\xi}\phi_{i,1,\xi'} + \tilde{\beta}e\varrho_i, \\
0 &= x_{i+1,m,\xi}\phi_{i+1,m,\xi} + y_{i,m,\xi}\phi_{i,m,\xi} + w_{i-1,m,\xi}\phi_{i-1,m,\xi} + (m-1)z_{i,m-1,\xi}\phi_{i,m-1,\xi} + \iota_{\xi'\to\xi}\phi_{i,m,\xi'}, \\
&\text{for} \quad m = 2,\dots,\overline{m}.
\end{aligned}$$

The system of equations can be written in matrix notation as $\mathbf{A}^T\phi + \tilde{\beta}e(\varrho \otimes \mathbf{1}\{m=1\}) = 0$, where $\mathbf{A}$ is the matrix defined in Section C.1 and $\phi = \text{vec}(\phi_{i,m,\xi})$. Finding the stationary probability mass function reduces to solving

$$\phi = -(\mathbf{A}^T)^{-1}\tilde{\beta}e(\varrho \otimes \mathbf{1}\{m=1\}), \tag{C.9}$$

and normalizing the result to ensure the regularity condition

$$\sum_{\xi\in\{\xi_L,\xi_H\}}\sum_{m=1}^{\overline{m}} m \sum_{i=1}^{I} \phi_{i,m,\xi}\Delta n = 1$$

holds.